Computing Hilbert Transform and Spectral Factorization for Signal Spaces of Smooth Functions

Holger Boche Volker Pohl
Technical University of Munich
Department of Electrical and Computer Engineering
Chair of Theoretical Information Technology

45th IEEE International Conference on Acoustics, Speech, and Signal Processing (ICASSP)
May 5, 2020
Subject and Outline of the Talk

Is it always possible to calculate the Hilbert transform and the spectral factorization on a digital computer?

Outline

1. Hilbert Transform and Spectral Factorization – A very short Introduction
2. Review of Computability Theory
3. Non-Computability/Computability of the Hilbert Transform and Spectral Factorization
4. Summary and Outlook
Hilbert Transform and Spectral Factorization
The Hilbert Transformation

Let \( f \in L^2(\partial \mathbb{D}) \) be function on the unit circle \( \partial \mathbb{D} = \{ z \in \mathbb{C} : |z| = 1 \} \) with Fourier series

\[
f(e^{i\theta}) = \sum_{n \in \mathbb{Z}} c_n(f) e^{in\theta}\quad \text{with} \quad c_n(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\tau}) e^{-in\tau} \, d\tau
\]

With \( f \) one associates its conjugate function, defined by

\[
\tilde{f}(e^{i\theta}) = (Hf)(e^{i\theta}) = -i \sum_{n \in \mathbb{Z}} \text{sgn}(n) c_n(f) e^{in\theta}\quad \text{with} \quad \text{sgn}(n) = \begin{cases} 0 & : n = 0 \\ n/|n| & : n \neq 0 \end{cases}.
\]

The linear mapping \( H : f \mapsto \tilde{f} \) is known as Hilbert transform and can be written as a principal value integral as

\[
\tilde{f}(e^{i\theta}) = (Hf)(e^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f(e^{i\tau})}{\tan\left(\frac{\theta - \tau}{2}\right)} \, d\tau, \quad \theta \in [-\pi, \pi).
\]

Application

- Physics: Kramers–Kronig relations
- Real- and imaginary part of a causal signal are related by the Hilbert transform
Spectral Factorization

Let $\phi$ be a spectral density. That is
- a non-negative real function on the unit circle $\partial \mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$
- satisfying the Paley–Wiener (Szegö) condition $\log \phi \in L^1(\partial \mathbb{D})$

Spectral factorization is the operation of writing $\phi$ as

$$\phi(e^{i\omega}) = \phi_+(e^{i\omega}) \phi_-(e^{i\omega}) = |\phi_+(e^{i\omega})|^2, \quad \omega \in [-\pi, \pi).$$

with the spectral factor $\phi_+$ and its para-Hermitian conjugate $\phi_-(z) = \overline{\phi_+(1/z)}$ for $z \in \mathbb{C}$.

The spectral factor $\phi_+$ is an outer function (a "minimum-phase system"), i.e.
- $\phi_+(z)$ is analytic for every $z \in \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$
- $\phi(z) \neq 0$ for all $z \in \mathbb{D}$.

The spectral factor can be written as

$$\phi_+(z) = (S\phi)(z) = \exp\left(\frac{1}{4\pi} \int_{-\pi}^\pi \log \phi(e^{i\omega}) \frac{e^{i\omega} + z}{e^{i\omega} - z} \, d\omega\right), \quad z \in \mathbb{D}.$$

We call the mapping $S : \phi \mapsto \phi_+$ the spectral factorization mapping.

Applications

- Wiener–Kolmogorov theory of smoothing and prediction of stationary time series
- causal Wiener filter: Communications, signal processing, control theory, · · ·

Volker Pohl (TUM) | Can every analog system be simulated on a digital computer? | ICASSP 2020
Computability
Computability – Intuition

- The true spectral factor $\phi_+$ is usually not known explicitly.
- A function $\phi_+$ is computable if it can be approximated effectively by a function $p_M$ which can perfectly be calculated on a digital computer.
  - $p_M$ might be a rational polynomial of a certain degree $M$
  - effective approximation $\Rightarrow$ control of approximation error

Computability (an informal definition)

The spectral factor $\phi_+$ is computable if there exists an algorithm with the following properties

- It can be implemented on a digital computer (a Turing machine).
- It has two inputs: 1. the spectral density $\phi$ 2. an error bound $\varepsilon > 0$.
- It is able to determine in finitely many steps an approximation $p_M$ of $\phi_+$ such that the true $\phi_+$ is guaranteed to be close to $p_M$, i.e. such that

$$\phi_+ \in \{ \psi \in X : \| \psi - p_M \|_X < \varepsilon \}$$

where $X$ is an appropriate Banach space with a corresponding norm $\| \cdot \|_X$. 

Volker Pohl (TUM) | Can every analog system be simulated on a digital computer? | ICASSP 2020
Computable Rational Numbers

Definition: A sequence \( \{r_n\}_{n \in \mathbb{N}} \subset \mathbb{Q} \) of rational numbers is said to be computable if there exist recursive functions \( a, b, s : \mathbb{N} \rightarrow \mathbb{N} \) with \( b(n) \neq 0 \) and such that

\[
    r_n = (-1)^s(n) \frac{a(n)}{b(n)}, \quad n \in \mathbb{N}.
\]

A recursive function \( a : \mathbb{N} \rightarrow \mathbb{N} \) is a mapping that is build form elementary computable functions and recursion and can be calculated on a Turing machine.

Turing machine

• can simulate any given algorithm and therewith provide a simple but very powerful model of computation.
• is a theoretical model describing the fundamental limits of any realizable digital computer.
• Most powerful programming languages are called Turing-complete (such as C, C++, Java, etc.).
Computable Real Numbers

▷ Any real number \( x \in \mathbb{R} \) is the limit of a sequence of rational numbers.
▷ For \( x \in \mathbb{R} \) to be computable, the convergence has to be effective.

**Definition (Computable number):** A real number \( x \in \mathbb{R} \) is said to be *computable* if there exists a computable sequence \( \{r_n\}_{n \in \mathbb{N}} \subset \mathbb{Q} \) of rational numbers which *converges effectively* to \( x \), i.e. if there exists a recursive function \( e : \mathbb{N} \to \mathbb{N} \) such that for all \( N \in \mathbb{N} \)
\[
|x - r_n| \leq 2^{-N} \quad \text{whenever} \quad n \geq e(N).
\]
\( \Rightarrow \) \( x \in \mathbb{R} \) is computable if a Turing machine can approximate it with exponentially vanishing error.

- \( \mathbb{R}_c \) stand for the set of all *computable real numbers*.
- \( \mathbb{C}_c = \{x + iy : x, y \in \mathbb{R}_c\} \) stands for the set of all *computable complex numbers*.
- Note that the set of computable numbers \( \mathbb{R}_c \subset \mathbb{R} \) is only *countable*. 
**Computable Functions**

**Definition:** A function \( f : \partial \mathbb{D} \to \mathbb{R} \) on the unit circle is said to be computable if

(a) \( f \) is Banach–Mazur computable, i.e. if \( f \) maps computable sequences \( \{x_n\}_{n \in \mathbb{N}} \subset \mathbb{R}_c \) onto computable sequences \( \{f(x_n)\}_{n \in \mathbb{N}} \subset \mathbb{R}_c \).

(b) \( f \) is effective uniformly continuous, i.e. if there is a recursive function \( d : \mathbb{N} \to \mathbb{N} \) such that for every \( N \in \mathbb{N} \) and all \( \zeta_1, \zeta_2 \in \partial \mathbb{D} \) with \( |\zeta_1 - \zeta_2| \leq 1/d(N) \) always \( |f(\zeta_1) - f(\zeta_2)| \leq 2^{-N} \) is satisfied.

**Lemma (equivalent definition of computability):**
A function \( f : \partial \mathbb{D} \to \mathbb{R} \) on the unit circle is computable if and only if there exists a sequence of rational polynomials \( \{p_m\}_{m \in \mathbb{N}} \) which converges effectively to \( f \) in the uniform norm, i.e. if there exists a recursive function \( e : \mathbb{N} \to \mathbb{N} \) such that for all \( \theta \in (-\pi, \pi) \) and every \( N \in \mathbb{N} \)

\[
m \geq e(N) \quad \text{implies} \quad |f(e^{i\theta}) - p_m(e^{i\theta})| \leq 2^{-N}.
\]

**Remark:**
- There exist various notions of computability e.g. Borel- or Markov computability.
- Banach–Mazur computability is the weakest form of computability.

\( \Rightarrow \) If a function is not Banach–Mazur computable then it is not computable with respect to any other notion of computability.
Computable Functions in Banach Spaces

We consider functions in a Banach space $X$ of functions on $\partial \mathbb{D}$ with norm $\|f\|_X$.

**Definition:** A function $f \in X$ is said to be $X$-computable if

(a) $f$ is computable (i.e. effectively approximable by rational polynomials $p_m$).

(b) its norm $\|f\|_X$ is computable $\Rightarrow$ $\|f - p_m\|_X$ converges to zero effectively as $m \to \infty$.

The set of all $X$-computable functions is denoted by $X_c$.

---


Non-Computability and Computability of the Hilbert Transformation
The Hilbert Transform is Generally not Computable

There exists computable continuous functions such that its Hilbert transform is not a computable function.

- **Dirichlet energy**: For every $f \in C(\partial D)$ with $f(e^{i\theta}) = \sum_{n \in \mathbb{Z}} c_n(f) e^{in\theta}$ define
  $$\|f\|_E = \left( \sum_{n \in \mathbb{Z}} |n| |c_n(f)|^2 \right)^{1/2}.$$  

- **Signal space**: Continuous functions of finite Dirichlet energy
  $$\mathcal{B} = \{ f \in C(\partial D) : \|f\|_E < \infty \} \quad \text{with norm} \quad \|f\|_{\mathcal{B}} = \max(\|f\|_{\infty}, \|f\|_E).$$

**Theorem**: There exist computable functions $f \in \mathcal{B}_c$ so that
  1. $\tilde{f} \in \mathcal{B}$ with $0 < \tilde{f}(0) < 1$ is absolute continuous
  2. $f$ is absolute continuous
  3. $f \in \mathcal{W}$ with $\|f\|_{\mathcal{W}} < 1$

but such that $\tilde{f}(0) = (Hf)(0) \notin \mathbb{R}_c$.  

Sets of Computable Hilbert Transforms

**Questions:** What are sufficient conditions of $f \in C_c(\partial D)$ so that $\tilde{f}$ is computable?

**Answer:** $f' \in L^p_c(\partial D)$ with $p > 1$.

---

**Theorem:**
Let $f \in C_c(T)$ be absolute continuous so that there exists an $p \in \mathbb{R}_c$, $p > 1$ such that $f' \in L^p_c(\partial D)$.
Then $\tilde{f} = Hf$ is a computable continuous function, i.e. $\tilde{f} \in C_c(\partial D)$.

**Theorem:**
There exists an absolute continuous $f \in C_c(\partial D)$ with $f' \in L^1_c(\partial D)$ so that $f \in \mathcal{W}$ and $\tilde{f} \in \mathcal{W}$
but such that $\tilde{f}(0) = (Hf)(0) \notin \mathbb{R}_c$. 
Non-Computability and Computability of the Spectral Factorization
Spectral Densities

We are going to show that the spectral factor

$$\phi_+(z) = (S\phi)(z) = \exp\left(\frac{1}{4\pi} \int_{-\pi}^{\pi} \log \phi(e^{i\theta}) \frac{e^{i\theta} + z}{e^{i\theta} - z} d\tau\right), \quad z \in \mathbb{D}$$

is not computable, even for computable spectral densities $\phi$ with very nice properties.

**Definition (Set $\mathcal{D}$ of nice spectral densities):**

A spectral density $\phi \in \mathcal{C}(\partial \mathbb{D})$ is said to belong to the set $\mathcal{D}$, if it has the following properties:

- $\phi$ is absolute continuous.
- $\phi$ is strictly positive on $\partial \mathbb{D}$, i.e. $\min_{\zeta \in \partial \mathbb{D}} \phi(\zeta) = s > 0$.
- $\phi$ belongs to the Wiener algebra $\mathcal{W}$, i.e. $\phi$ possess an absolutely converging Fourier series
- $\phi$ has finite Dirichlet energy, i.e. $\|\phi\|_E < \infty$
- The spectral factor $\phi_+$ has the same properties as $\phi$, i.e. $\phi_+$ is absolute continuous, in the Wiener algebra $\mathcal{W}$, and has finite Dirichlet energy.
The Non-Computability of the Spectral Factorization

**Theorem:**
To every computable point \( \zeta \in \partial \mathbb{D} \) on the unit circle, there exists a computable spectral density \( \phi \in \mathcal{D} \) such that \( \phi_+(\zeta) \) is not a computable number, i.e. such that \( \phi_+(\zeta) \notin \mathbb{C}_c \).

**Remark:**
- \( \phi_+(\zeta) \) is not a computable number \( \Rightarrow \phi_+ \) is not Banach-Mazur computable.
- So \( \phi_+ \) is not computable in any stronger notion of computability.
- Note that the input, i.e. the spectral density \( \phi \) is computable. However, the corresponding spectral factor \( \phi_+ \) might not be computable.

---

Computability of Spectral Factorization

**Questions:** What are sufficient conditions of \( \phi \in \mathcal{C}_c(\partial D) \) so that \( \phi_+ \) is computable?

**Answer:** \( \phi' \in L^p_c(\partial D) \) with \( p > 1 \).

---

**Theorem:**
Let \( \phi \in \mathcal{C}_c(\mathbb{T}) \) be strictly positive on \( \partial D \) so that there exists an \( p \in \mathbb{R}_c, p > 1 \) such that \( \phi' \in L^p_c(\partial D) \). Then \( \phi_+ \) is a computable continuous function, i.e. \( \phi_+ \in \mathcal{C}_c(\partial D) \).

**Theorem:**
There exists a strictly positive spectral density \( \phi \in \mathcal{C}_c(\partial D) \) with \( \phi' \in L^1_c(\partial D) \) so that \( \phi_+(1) \notin \mathbb{R}_c \).
Summary

▷ There is no closed form expression for the Hilbert transform $Hf$ or the spectral factor $\phi_+$.
   \[ \implies \text{Numerically approximation methods (on digital computers) are applied to determine } Hf \text{ or } \phi_+. \]

▷ Numerically approximation:
   Given $f$ or $\phi$ and $\varepsilon > 0$, determine (in finite time) a confidence interval of width $2\varepsilon$ in which the (unknown) $\tilde{f}$ or $\phi_+$, respectively, lies. \[ \Rightarrow \tilde{f} \text{ or } \phi_+ \text{ is computable.} \]

▷ Negative results:
   − There exist computable continuous functions $f$ with very good analytic properties (finite energy, absolute continuous, etc.) for $\tilde{f} = Hf$ is not computable.
   − There exist computable spectral densities $\phi$ with very decent analytic properties (finite energy, absolute continuous, etc.) for which the spectral factor $\phi_+$ is not computable.

▷ Positive results:
   − Sharp characterization of sets of functions $f$ and spectral densities $\phi$ such that $\tilde{f}$ and $\phi_+$ is computable.