COMPACT REPRESENTATION OF INTERVAL GRAPHS OF BOUNDED DEGREE AND CHROMATIC NUMBER

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Set up

A simple undirected graph $G$ is called an Interval graph if (a) with every vertex we can associate a closed interval on the real line, and (b) two vertices share an edge if and only if the corresponding intervals are not disjoint.
Set up

Given a set $T$ consisting of combinatorial objects with certain property, a data structure $Z$ is called

- Succinct if $Z$ can store any arbitrary member $x$ from $T$ using $\log(|T|) + o(\log(|T|))$ bits, OR
- Compact if $Z$ can store any arbitrary member $x$ from $T$ using $O(\log(|T|))$ bits, along with fast query support.

There already exist succinct/compact data structures for various combinatorial objects like arbitrary graphs, planar graphs, trees, deterministic finite automata, permutations, equivalence classes and many more.
**Prior Work**

- Acan et al. (Algorithmica 2021) proposed a succinct data structure for storing and navigating interval graphs.
- More specifically, given an unlabeled interval graph \( G \) with \( n \) vertices, they first show that at least \( n \log n - 2n \log \log n - O(n) \) bits are necessary to represent \( G \).
- This is followed by a matching data structure consuming \( n \log n + O(n) \) bits of space with constant time queries i.e., degree, adjacency and neighborhood.

Can we design efficient data structures whose space consumption beats the information-theoretic lower bound under some bounded parameter condition?
Our Main Results

• When the maximum degree of any interval graph is bounded by $k$, we show that there exists an $(n \log k + O(n))$-bit data structure. Thus, our data structure surpasses the information-theoretic lower bound when $k = O(n^\epsilon)$ where $0 < \epsilon < 1$.

• We augment the upper bound result by giving an explicit $((1/6) n \log k - O(n))$-bit enumerative lower bound. Our result provides counting lower bound by taking into consideration maximum degree as a parameter for interval graphs for the first time in the related literature.

• Finally, we consider interval graphs with bounded chromatic number $p$, and here, we design a $(p-1)n + o(pn)$ bit data structure with efficient navigational query support. Thus, our data structure surpasses the information-theoretic lower bound when $p = o(\log n)$. 
Upper Bound

This upper bound follows from the result of Acan et al.’s (Algorithmica 2021) result in a straightforward manner.

Every interval has distinct start and end point. Overall, for $n$ intervals, all the endpoints make up $2n$ distinct integers from 1 to 2n without loss of generality.

$S$ is a bit string of size $2n$ bits with 0s in starting locations and 1 at the ending locations.

$r$ stores difference between end point and start point of the intervals starting from left to right.

With additional $o(n)$ bit structures (Rank and Select), it is possible to store $r$ and $S$ using $n \log n + O(n)$ bits such that degree/adjacency/neighborhood queries can be supported.

As the maximum degree of our input interval graph is bounded by $k$, in total the data structure consumes $(n \log k + O(n))$-bit along with supporting fast queries.

Is this optimal?
Lower Bound

For any interval graph $G$ with $n$ vertices, if the maximum degree of $G$ is $k$, at least $((1/6) n \log k - O(n))$-bits are necessary to represent $G$.

Proof Idea:

- Let $T$ be a set of all non-isomorphic interval graphs with $n$ vertices where for each graph $G$ in $T$, its interval representation satisfies (i) all the starting and endpoints of the intervals are distinct, and (ii) the maximum degree of $G$ is at most $k$. Then $|T|$ gives our desired lower bound.

- We first obtain an interval representation from the interval graph using bundle hypergraph.

- For an interval graph $G$, let $C$ be a set of all the inclusion-maximal cliques in $G$. Also, let $B_v$ be the bundle at vertex $v$, which is a set of maxcliques in $C$ containing $v$. Then the bundle hypergraph $\Delta = (C, E(\Delta))$ is a hypergraph where its hyperedges are the bundles of $G$. i.e., $E(\Delta) = \{B_v \mid v \in V\}$. 
Proof Idea

• It is known that if $G$ is an interval graph, one can define an ordering among the maxcliques in $C$ to satisfy the property that every hyperedge of $\Delta$ consists of consecutive maxcliques in $C$. Thus, by denoting the $i$-th maxclique in $C$ as $C_i$, one can define the interval representation of $G$ as for each $v$ as $I_v = [i, j]$ if $B_v = \{C_i, C_{i+1}, \ldots, C_j\}$.

• Note that multiple intervals can share same end points, but we can easily make the endpoints distinct by changing the shared endpoints as consecutive integers.

• We say $B_u$ and $B_v$ overlap if $B_u \cap B_v \neq \{B_u, B_v, \emptyset\}$. Similarly, $\Delta$ is called overlap connected if for any two hyperedges $B_u$ and $B_v$, there exists a sequence $S$ of hyperedges from $B_u$ to $B_v$ where any two consecutive hyperedges in $S$ are overlapped.
Proof Idea

(Kobler et al. SICOMP 2011) showed the following

- there exists a bijection between the set of all non-isomorphic interval graphs and the set of all non-isomorphic bundle hypergraphs, and
- if $\Delta$ is overlap connected, there exist at most two minimal interval representations of $\Delta$, $C$ and $D$, where $D$ is a mirror image of $C$.

Thus, by counting the number of distinct minimal interval representation whose corresponding bundle hypergraph $\Delta$ is overlap connected, we can obtain the lower bound of the number of non-isomorphic interval graphs.

$((1/6) \, n \log k - O(n))$ bits are needed when maximum degree is bounded by $k$.

In the extended version of our paper, we show similar results can be obtained for circular-arc graphs when parameterized by maximum degree and chromatic number.
Conclusion

• All our data structures are compact. Can we make them succinct?

• Which parameter would give better compression than the ones we considered here?

• Systematic study of parameterized data structures for combinatorial objects.

Thank you for your attention