Computationally Efficient Waveform Design in Spectrally Dense Environment

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Introduction

- Recently in radar systems waveform design in spectrally dense environment [1] has aroused noticeable interest
- Solution methods exist for the problem (see e.g. [2], [3]) but they are computationally inefficient
- When radar system operates at GHz level radar code dimension becomes large, need for computationally efficient solution methods
- Here we develop new computationally efficient method to design transmitter waveform in spectrally dense environment
- New method is based on ADMM algorithm [4] alongside Majorization-Minimization step [5]

Problem formulation

- Similarly to [3], denote transmitted fast-time radar code vector by $\mathbf{c}$ and fast-time observation signal by $\mathbf{v}$:

$$\mathbf{c} = (c[1], c[2], ..., c[N])^T, \mathbf{v} = \alpha\mathbf{c}+\mathbf{n}, \;\; \mathbf{c}, \mathbf{v} \in \mathbb{C}^N, \alpha \in \mathbb{C} \quad (1)$$

- Matched filtering $\mathbf{v}$ with filter $\mathbf{h} \in \mathbb{C}^N$ yields $y = \mathbf{h}^H\mathbf{v}$. Write $y = y_s + y_n$, where $y_s = \alpha\mathbf{h}^H\mathbf{c}$ and $y_n = \mathbf{h}^H\mathbf{n}$. SINR is given as:

$$\text{SINR} = \frac{|y_s|^2}{|y_n|^2} = \frac{|\alpha|^2|\mathbf{h}^H\mathbf{c}|^2}{|\mathbf{h}^H\mathbf{n}|^2} = \frac{|\alpha|^2|\mathbf{h}^H\mathbf{c}|^2}{\mathbf{h}^H\underbrace{\mathbf{n}\mathbf{n}^H}_{=\mathbf{M}}\mathbf{h}} \quad (2)$$

- To maximize SINR w.r.t. $\mathbf{h}$, we choose $\mathbf{h} = \mathbf{M}^{-1}\mathbf{c}$, which yields $\text{SINR} = |\alpha|^2\mathbf{c}^H\mathbf{M}^{-1}\mathbf{c}$

Problem formulation

- Introduce constrained bandwidths $\{\Omega_k\}_{k\in\{1,2,\ldots,K\}}$, where $\Omega_k = \left[f_1^k, f_2^k\right]$. The energy $\mathbf{c}$ radiates to constrained bandwidths is (see e.g. [3]):

$$\sum_{k=1}^{K} w_k \int_{\Omega_k} |\mathcal{F}_{\mathbb{N}}\{\mathbf{c}\}|^2 df = \mathbf{c}^H \mathbf{R}_\text{I} \mathbf{c}, \tag{3}$$

where $\{w_k\}_{k=1}^K$ are non-negative weights, $\mathcal{F}_{\mathbb{N}}\{\mathbf{c}\}$ stands for the discrete-time Fourier transform of $\mathbf{c}$ given as $\mathcal{F}_{\mathbb{N}}\{\mathbf{c}\} \triangleq \sum_{k=1}^{N} c[k]e^{-j2\pi kf}$, and $\mathbf{R}_\text{I} \triangleq \sum_{k=1}^{K} w_k \mathbf{R}_\text{I}^k$ with $[\mathbf{R}_\text{I}^k]_{m,l} = (e^{j2\pi f_2^k(m-l)} - e^{j2\pi f_1^k(m-l)})/e^{j2\pi(m-l)}$, if $m \neq l$, and $[\mathbf{R}_\text{I}^k]_{m,l} = f_2^k - f_1^k$, if $m = l$.

Problem formulation

- If radar code energy $\|\mathbf{c}\|^2$ is unit constrained and required to be in similarity region with reference code $\mathbf{c}_0$ alongside radiation energy constraint $\mathbf{c}^H\mathbf{R}_\mathrm{I}\mathbf{c} \leq E_I$, SINR maximization problem can be written:

$$\mathcal{P}_1 : \begin{cases} \max\limits_{\mathbf{c}} & |\alpha|^2 \mathbf{c}^H\mathbf{M}^{-1}\mathbf{c} & \text{(4a)} \\ \text{s.t. :} & \|\mathbf{c}\|^2 = 1 & \text{(4b)} \\ & \mathbf{c}^H\mathbf{R}_\mathrm{I}\mathbf{c} \leq E_\mathrm{I} & \text{(4c)} \\ & \|\mathbf{c} - \mathbf{c}_0\|^2 \leq \epsilon & \text{(4d)} \end{cases}$$

Problem formulation

- $\mathcal{P}_1$ is equal to:

$$\mathcal{P}_1^{(1)} : \begin{cases} \min\limits_{\mathbf{c}} & -\mathbf{c}^H \mathbf{R} \mathbf{c} & \text{(5a)} \\ \text{s.t. :} & \|\mathbf{c}\|^2 = 1 & \text{(5b)} \\ & \mathbf{c}^H \mathbf{R}_\text{I} \mathbf{c} \leq E_\text{I} & \text{(5c)} \\ & \|\mathbf{c} - \mathbf{c}_0\|^2 \leq \epsilon & \text{(5d)} \end{cases}$$

where $\mathbf{c}, \mathbf{c}_0 \in \mathbb{C}^N$ and $\mathbf{R}_\text{I}, \mathbf{R} = \mathbf{M}^{-1} \in \mathbb{C}^{N \times N}$

Majorization-Minimization step

- Due to independence of real and imaginary components we can write $\mathbf{c}, \mathbf{c}_0, \mathbf{R}$ and $\mathbf{R}_\mathrm{I}$ as:

$$\mathbf{R} = \begin{bmatrix} \mathrm{Re}\{\mathbf{R}\} & -\mathrm{Im}\{\mathbf{R}\} \\ \mathrm{Im}\{\mathbf{R}\} & \mathrm{Re}\{\mathbf{R}\} \end{bmatrix}, \mathbf{c} = \begin{bmatrix} \mathrm{Re}\{\mathbf{c}\} \\ \mathrm{Im}\{\mathbf{c}\} \end{bmatrix} \text{ and } \mathbf{c}_0 = \begin{bmatrix} \mathrm{Re}\{\mathbf{c}_0\} \\ \mathrm{Im}\{\mathbf{c}_0\} \end{bmatrix}.$$

- Let us use use surrogate $\mathbf{Q} = \mu\mathbf{I} - \mathbf{R} \succeq 0$, $\mu > 0$ to upper-bound objective. We get real-valued optimization problem $\mathcal{P}_2$:

$$\mathcal{P}_2 : \begin{cases} \min\limits_{\mathbf{c}} & \mathbf{c}^T\mathbf{Q}\mathbf{c} & \text{(6a)} \\ \text{s.t. :} & \|\mathbf{c}\|^2 = 1 & \text{(6b)} \\ & \mathbf{c}^T\mathbf{R}_\mathrm{I}\mathbf{c} \leq E_\mathrm{I} & \text{(6c)} \\ & \|\mathbf{c} - \mathbf{c}_0\| \leq \epsilon & \text{(6d)} \end{cases}$$

where $\mathbf{c}, \mathbf{c}_0 \in \mathbb{R}^{2N}$ and $\mathbf{Q}, \mathbf{R}_\mathrm{I} \in \mathbb{R}^{2N \times 2N}$

Apply ADMM to $\mathcal{P}_2$

- To allow separability of $\mathbf{c}^T\mathbf{Q}\mathbf{c}$, let us introduce slack variable $\mathbf{z}$ with constraint $\mathbf{c} = \mathbf{z}$. Augmented Lagrangian $L_\rho(\mathbf{c}, \mathbf{z}, \boldsymbol{\lambda})$ for minimization problem $\min_{\mathbf{c}} \mathbf{c}^T\mathbf{Q}\mathbf{c}$ s.t.: $\mathbf{c} = \mathbf{z}$:

$$L_\rho(\mathbf{c}, \mathbf{z}, \boldsymbol{\lambda}) = \mathbf{c}^T\mathbf{Q}\mathbf{c} + \boldsymbol{\lambda}^T(\mathbf{c} - \mathbf{z}) + \frac{\rho}{2}\|\mathbf{c} - \mathbf{z}\|^2. \tag{7}$$

- ADMM-steps for $\mathcal{P}_2$:

$$\begin{cases} \mathbf{c}_{k+1} = \arg\min_{\mathbf{c}} L_\rho\left(\mathbf{c}, \mathbf{z}_k, \boldsymbol{\lambda}_k\right) & \text{(8a)} \\ \mathbf{z}_{k+1} = \arg\min_{\mathbf{z}} L_\rho\left(\mathbf{c}_{k+1}, \mathbf{z}, \boldsymbol{\lambda}_k\right) & \text{(8b)} \\ \boldsymbol{\lambda}_{k+1} = \boldsymbol{\lambda}_k + \rho\left(\mathbf{c}_{k+1} - \mathbf{z}_{k+1}\right), & \text{(8c)} \end{cases}$$

- Next $\mathbf{c}$-variable update and $\mathbf{z}$-variable update are solved.

c-variable update

- $\mathbf{c}$-variable update (8a) can be written as:

$$\mathbf{c}_{k+1} = \arg\min_{\mathbf{c}} L_\rho(\mathbf{c}, \mathbf{z}_k, \boldsymbol{\lambda}_k) = \arg\min_{\mathbf{c}} \left\{ \mathbf{c}^T \mathbf{Q}\mathbf{c} + (\boldsymbol{\lambda} - \rho\mathbf{z})^T \mathbf{c} \right\}$$
$$= \arg\min_{\mathbf{c}} h(\mathbf{c}) \quad | \text{s.t. } \|\mathbf{c}\|^2 = 1, \ \|\mathbf{c} - \mathbf{c}_0\|^2 \leq \epsilon. \qquad (9)$$

- Objective function $h(\mathbf{c})$ is continuously differentiable and $\nabla_{\mathbf{c}} h$ is L-Lipschitz continuous. To minimize $h(\mathbf{c})$ we use gradient descent:

$$\mathbf{c}_{k+1} = \mathbf{c}_k - \frac{1}{L}\left(\left(\mathbf{Q} + \mathbf{Q}^T\right)\mathbf{c}_k + (\boldsymbol{\lambda} - \rho\mathbf{z})\right), \qquad (10)$$

where Lipschitz constant can be found by noticing:

$$|\nabla_{\mathbf{c}} h(\boldsymbol{\kappa}) - \nabla_{\mathbf{c}} h(\mathbf{c})| = \left|\left(\mathbf{Q} + \mathbf{Q}^T\right)(\boldsymbol{\kappa} - \mathbf{c})\right| \leq L\,|\boldsymbol{\kappa} - \mathbf{c}|$$
$$\Rightarrow \left|\sum_{p=1}^{2N}\left(\mathbf{Q}_{[i,p]} + \mathbf{Q}^T_{[i,p]}\right)\right| \leq L, \forall i = 1, \cdots, 2N$$

c-variable update

- Gradient descent yields updated $\mathbf{c}$ that has $\|\mathbf{c}\|_2^2 \neq 1$ and possibly $\|\mathbf{c} - \mathbf{c}_0\| \geq \epsilon$.
- Denote $\Theta = \{\mathbf{c} \in \mathbb{R}^{2N} \mid \|\mathbf{c}\|^2 = 1$ and $\|\mathbf{c} - \mathbf{c}_0\|^2 \leq \epsilon$, for some $\mathbf{c}_0 \in \mathbb{R}^{2N}\}$
- Cheap way to project $\mathbf{c}$ back to unitary region is to divide updated $\mathbf{c}$ by its L^2-norm:

$$\hat{\mathbf{c}}_{k+1} = \mathbf{c}_{k+1} / \|\mathbf{c}_{k+1}\| \tag{11}$$

c-variable update

- Next $\hat{\mathbf{c}}_{k+1}$ is rotated to region Θ with steps introduced in Algorithm 1.

Algorithm 1: Rotate c toward c_0 as long as region $\|c - c_0\| \leq \epsilon$ is reached

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<u>function</u> RotateVector($\mathbf{c}, \mathbf{c}_0, \alpha', \epsilon$);

Input : $\mathbf{c}$, $\mathbf{c}_0$, α' and ϵ

Output : $\mathbf{c}$

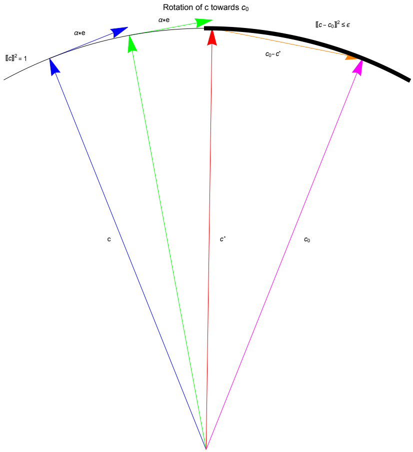
while $\|\boldsymbol{c} - \boldsymbol{c}_0\| > \epsilon$ **do**

$$\widetilde{\mathbf{c}} = \mathbf{c}_0 - \mathrm{proj}_{\mathbf{c}}(\mathbf{c}_0) = \mathbf{c}_0 - \frac{\mathbf{c}_0^H, \mathbf{c}}{\|\mathbf{c}\|^2}\mathbf{c};$$

$$\mathbf{e} = \frac{\widetilde{\mathbf{c}}}{\|\widetilde{\mathbf{c}}\|}, \ \mathbf{c}^* = \mathbf{c} + \alpha'\mathbf{e},$$

$$\mathbf{c} = \frac{\mathbf{c}^*}{\|\mathbf{c}^*\|};$$

end



c-variable update

- The combination of steps (10), (11) and Algorithm 1 can be shown to be solution steps to projected gradient step for problem $\min_{\mathbf{c}} h(\mathbf{c})$ subject to $\mathbf{c} \in \Theta$:

$$\begin{cases} \mathbf{y}_{k+1} = \mathbf{c}_k - \frac{1}{L}\nabla h(\mathbf{c}_k) & \text{(12a)} \\ \mathbf{c}_{k+1} = \min_{\mathbf{c}\in\Theta} \left\| \mathbf{y}_{k+1} - \mathbf{c} \right\|. & \text{(12b)} \end{cases}$$

- By using angular coordinates $\phi \in \mathbb{R}^{2N-1}$ step (12b) can be written as:

$$\begin{cases} \phi_{k+1} = \arg\min_{\phi\in\Omega} \|\phi^* - \phi\| & \text{(13a)} \\ \mathbf{c}_{k+1} = \mathbf{c}(\phi_{k+1}). & \text{(13b)} \end{cases}$$

where $\Omega = \left\{\phi \in \mathbb{R}^{2N-1} \mid \|\mathbf{c}(\phi) - \mathbf{c}_0(\phi)\|^2 \leq \epsilon\right\}$ and $\phi^* = \arg\min_{\phi} h(\mathbf{c}(\phi))$.

z-variable update

- $\mathbf{z}$-variable update (8b) can be written as:

$$\begin{aligned}
\mathbf{z}_{k+1} &= \arg\min_{\mathbf{z}} L_{\rho}\left(\mathbf{c}_{k+1}, \mathbf{z}, \boldsymbol{\lambda}_k\right) \\
&= \arg\min_{\mathbf{z}} \left\{ \boldsymbol{\lambda}^T(\mathbf{c} - \mathbf{z}) + \frac{\rho}{2}\|\mathbf{c} - \mathbf{z}\|^2 \right\} \\
&= \arg\min_{\mathbf{z}} \left\{ \left\| \mathbf{z} - (\mathbf{c} + \frac{1}{\rho}\boldsymbol{\lambda}) \right\|^2 \right\} |\text{s.t. } \mathbf{z}^T\mathbf{R}_{\text{l}}\mathbf{z} \leq E_{\text{l}}. \quad (14)
\end{aligned}$$

- Lagrangian for (14) is given as:

$$L(\mathbf{z}, \gamma) = \left\| \mathbf{z} - (\mathbf{c} + \frac{1}{\rho}\boldsymbol{\lambda}) \right\|^2 + \gamma(\mathbf{z}^T\mathbf{R}_{\text{l}}\mathbf{z} - E_{\text{l}}). \quad (15)$$

z-variable update

- Karush-Kuhn-Tucker (KKT) conditions for the minimization problem (14):

$$\begin{cases} \nabla_{\mathbf{z}} L(\mathbf{z}^*, \gamma^*) = 0 & \text{(16a)} \\ \gamma^* \geq 0 & \text{(16b)} \\ \gamma^*((\mathbf{z}^*)^T \mathbf{R}_l \mathbf{z}^* - E_l) = 0 & \text{(16c)} \\ (\mathbf{z}^T \mathbf{R}_l \mathbf{z} - E_l) \leq 0 & \text{(16d)} \\ \nabla_{\mathbf{zz}} L(\mathbf{z}^*, \gamma^*) \succeq 0, & \text{(16e)} \end{cases}$$

- By (16a) and (16c):

$$\nabla_{\mathbf{z}} L(\mathbf{z}^*, \gamma^*) = 0 \Rightarrow (\mathbf{I} + \gamma^* \mathbf{R}_l)\, \mathbf{z}^* = \mathbf{c} + \frac{1}{\rho} \boldsymbol{\lambda}, \tag{17}$$

$$(\mathbf{z}^*)^T \mathbf{R}_l \mathbf{z}^* - E_l = 0, \tag{18}$$

where $\mathbf{z}^*$ and γ^* denotes critical points of Lagrangian $L(\mathbf{z}, \gamma)$.

z-variable update

- Now (17) can be written as iteration step (19):

$$\mathbf{z}_{k+1} = (\mathbf{I} + \gamma_{k+1}\mathbf{R}_\mathrm{l})^{-1}\left(\mathbf{c} + \frac{1}{\rho}\boldsymbol{\lambda}\right)$$
$$= \left(\mathbf{I} + \sum_{i=1}^{2N} \frac{\gamma_{k+1}\sigma_i}{1+\gamma_{k+1}\sigma_i}\mathbf{p}_i\mathbf{p}_i^T\right)\left(\mathbf{c} + \frac{1}{\rho}\boldsymbol{\lambda}\right). \qquad (19)$$

- $\gamma_{k+1} > 0$ can be found as the solution to (18):

$$\mathbf{z}_{k+1}^T\mathbf{R}_\mathrm{l}\mathbf{z}_{k+1} = E_\mathrm{l} \Leftrightarrow \sum_{i=1}^{2N} \frac{a_i\sigma_i}{(1+\gamma\sigma_i)^2} - E_\mathrm{l} = 0 \qquad (20)$$

where $a_i = (\mathbf{p}_i^T(\mathbf{c} + \frac{1}{\rho}\boldsymbol{\lambda}))^2$, σ_i is i'th eigenvalue and $\mathbf{p}_i$ corresponding eigenvector of $\mathbf{R}_\mathrm{l}$. Equation (20) can be efficiently solved by using Newton's method.

Proposed algorithm

- Collect c and z-variable updates to get final algorithm:

Algorithm 2: MM-algorithm

<u>function</u> MM$(\mathbf{Q}, \mathbf{c}_0, \mathbf{R}_I, E_I, \epsilon, K')$;

Input : $\mathbf{Q} = \mu\mathbf{I} - \mathbf{R} \succeq 0$, $\mathbf{c}_0$, $\mathbf{R}_I$, E_I, ϵ and K'

Output : $\mathbf{c}$

Initialize $\mathbf{c}$, $\mathbf{z}$ and $\boldsymbol{\lambda}$;

for $k = 1, k \leq K', k$ ++ **do**

$$\hat{\mathbf{c}}_{k+1} = \mathbf{c}_k - \frac{1}{L}\left(\left(\mathbf{Q} + \mathbf{Q}^T\right)\mathbf{c}_k + (\boldsymbol{\lambda} - \rho\mathbf{z})\right);$$

$$\widetilde{\mathbf{c}}_{k+1} = \frac{\hat{\mathbf{c}}_{k+1}}{\|\hat{\mathbf{c}}_{k+1}\|};$$

$$\mathbf{c}_{k+1} = \text{RotateVector}(\widetilde{\mathbf{c}}_{k+1}, \mathbf{c}_0, \alpha, \epsilon);$$

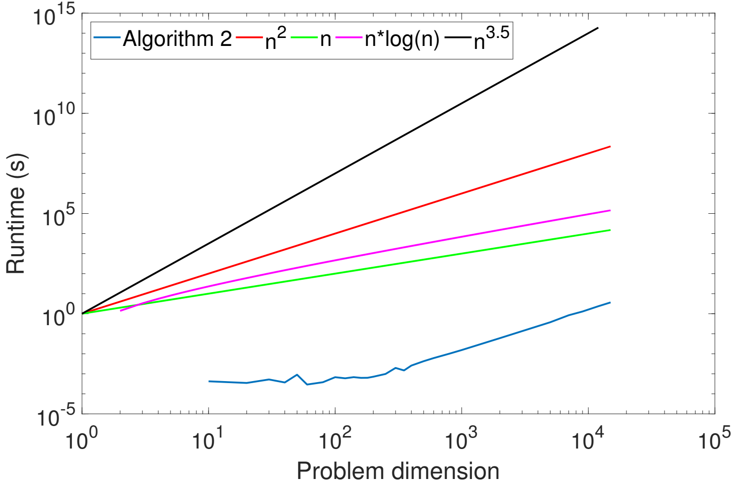
Solve $\sum_{i=1}^{2N} \frac{a_i\sigma_i}{(1+\gamma\sigma_i)^2} - E_I = 0$ for $\gamma_{k+1} > 0$;

$$\mathbf{z}_{k+1} = \left(\mathbf{I} + \sum_{i=1}^{2N} \frac{\gamma_{k+1}\sigma_i}{1+\gamma_{k+1}\sigma_i}\mathbf{p}_i\mathbf{p}_i^T\right)\left(\mathbf{c} + \frac{1}{\rho}\boldsymbol{\lambda}\right);$$

$$\boldsymbol{\lambda}_{k+1} = \boldsymbol{\lambda}_k + \rho(\mathbf{c}_{k+1} - \mathbf{z}_{k+1});$$

end

Time-Complexity graph



Simulation example

- Let us use Algorithm 2 in example environment. Consider radar with bandwidth of 6 GHz to be sampled at sampling frequency of $f_s = 12\text{GHz}$.
- Fast-time radar code has length $T = 1\mu s$ (i.e. $N = 12000$).
- The radar operates in spectrally busy environment with seven constrained bandwidths $\{\Omega_k\}_{k=1}^{7} = \{[0.0000,\ 0.0617], [0.0700,\ 0.1247], [0.1526,\ 0.2540], [0.3086,\ 0.3827], [0.4074,\ 0.4938], [0.6185,\ 0.7600], [0.8200,\ 0.9500]\}$.
- Covariance matrix is modelled as:

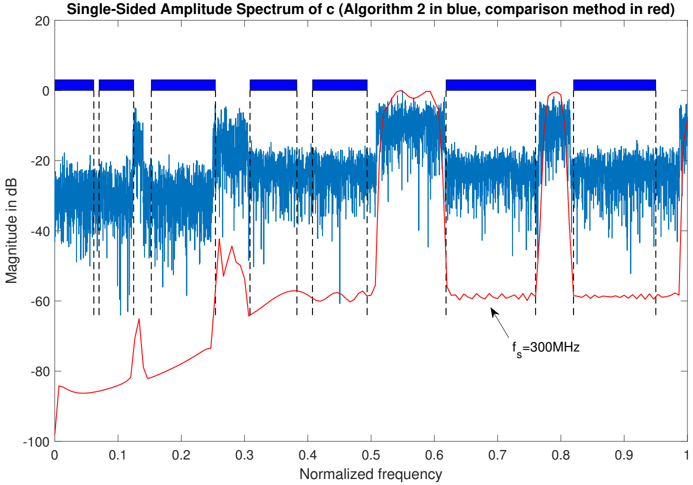
$$\mathbf{M} = \sigma_0 \mathbf{I} + \sum_{k=1}^{K} \frac{\sigma_{I,k}}{\Delta f_k} \mathbf{R}_I^k + \sum_{k=1}^{K_J} \sigma_{J,k} \mathbf{R}_{J,k} \tag{21}$$

- For reference signal we use linearly modulated signal $\mathbf{c}_0 = e^{j2\pi(f_\Delta t + f_0)t}$, with carrier frequency $f_0 = 1.8\text{GHz}$ and frequency range $f_\Delta = 3.6\text{GHz}/\mu s$.

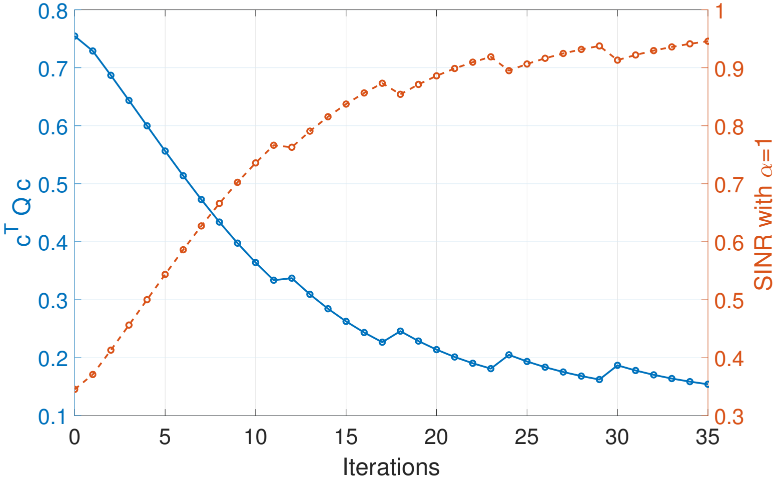
Simulation example

- $\sigma_0 = 0$dB (thermal noise level)
- $K = 7$ (number of licensed radiators)
- $\sigma_{I,k} = 10\text{dB}, \forall k \in \{1, ..., K\}$ (energy of coexisting telecom network operating on normalized frequency band $\Omega_k = [f_1^k, f_2^k]$)
- $\Delta f_k = f_2^k - f_1^k, \forall k \in \{1, ..., K\}$ (bandwidth associated with the k'th licensed radiator)
- $K_J = 2$ (number of active and unlicensed narrowband jammers)
- $\sigma_{J,k} = \begin{cases} 50\text{dB}, & k = 1 \\ 40\text{dB}, & k = 2, \end{cases}$ (energy of active jammers)
- $\mathbf{R}_{J,k} = \mathbf{r}_{J,k}\mathbf{r}_{J,k}^H, k = 1, ..., K_J$ (normalized disturbance covariance matrix of the k'th active unlicensed jammer)
- $\mathbf{r}_{J,k} = e^{j2\pi f_{j,k} n/f_s}, f_{J,1}/f_s = 0.7$ and $f_{J,2}/f_s = 0.75$
- $w_k = 1, \forall k \in \{1, ..., 7\}$ (weights in $\mathbf{R}_I$).

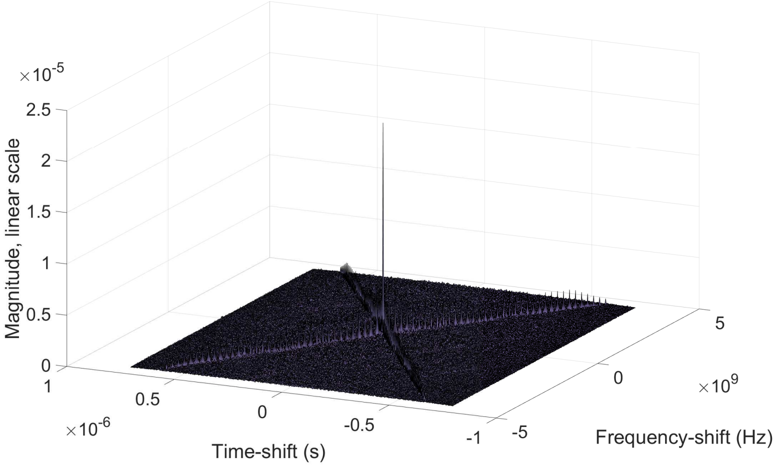
Frequency spectrum and comparison to other method [3]



SINR convergence



Ambiguity function



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