Estimating Centrality Blindly from Low-pass Filtered Graph Signals

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Motivation

Graphs are useful for describing the geometric structures of data from numerous fields, including social, energy, transportation, and neuronal networks.
Motivation

- Importance of nodes in the graph ➞ node centrality
- E.g., social network – most influential individuals, transport network – cities with largest population mobility.

**Research Question:** *Can we learn node centrality from data?*
Prior works

- **Graph learning:** recover the complete topology
  - *Statistical/physical models* – GMRF [Friedman et al., 2008], dynamical systems / causation [Shen et al., 2017].
  - *Graph signal processing (GSP) models* – smoothness of graph signals [Dong et al., 2016], inference with structural constraints [Egilmez et al., 2017], spectral template [Segarra et al., 2017].
  - and many others...

- **This work:** learning graph features **without the graph**
  - *Community inference* – blind community detection [Wai et al., 2018], Bayesian learning [Hoffmann et al., 2018], recovery from multi-graph [Roddenberry et al., 2020].
  - *Centrality learning* – centrality ranking inference [Roddenberry and Segarra, 2019], and this work.
Contribution

- We show that the folklore heuristic based on PCA works if data is generated from a ‘strong’ low-pass filter.

- For data generated from a ‘weak’ low-pass filter, we propose a boosting method with provably better estimation quality.

- Numerical experiments on synthetic and stocks data.
Graph Model and Centrality Measure

- **Undirected** graph $G = (V, E, A)$ with $V = \{1, \ldots, N\}$, symmetric adjacency matrix $A \in \mathbb{R}^{N \times N}$
- The adjacency matrix admits an eigenvalue decomposition (EVD) as $A = V \Lambda V^T$ s.t. $V$ = orthogonal, $\Lambda$ = Diag($\lambda_1, \ldots, \lambda_N$).
- Centrality is given by the eigen-centrality

$$ c_{eig} := \text{TopEV}(A) = v_1 $$

Network  

Adjacency Matrix
Graph Signal Model

- Observed data $y^t$ is produced by an excitation $x^t$ to be ‘processed’ by a graph filter $H(A)$

$$y^t = H(A)x^t$$
**Graph Filter**

- The graph filter $\mathcal{H}(A)$ is a matrix polynomial:
  
  $$\mathcal{H}(A) = \sum_{t=0}^{T-1} h_t A^t$$

- Set $h(\lambda) := \sum_{t=0}^{T-1} h_t \lambda^t$.

- Assume 1-low pass $\mathcal{H}(A)$:
  
  $$\max_{j=2,\ldots,N} |h(\lambda_j)|/|h(\lambda_1)| =: \eta < 1$$

  - $\eta \ll 1 \implies$ strong low-pass.
  - $\eta \approx 1 \implies$ weak low-pass.

- E.g., diffusion, op. dynamics.

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**Excitation Signal**

- The input $x^t$ is controlled by an external source $z^t$:
  
  $$x^t = Bz^t$$

- Assume a sparse influence matrix $B \in \mathbb{R}^{N \times k}$, $(k < N)$.

- E.g., influence from external source $z^t$ are localized to specific nodes on graph.
Blind centrality estimation

- Idea: apply PCA on the covariance of filtered graph signals, use the principal eigenvector as an estimate for $c_{eig}$

\[ Y = [y_1 \ldots y_m] \]

Observation

\[
C_y = \frac{1}{m} \sum_{t=1}^{m} y_t (y_t)^T
\]

Sample Covariance

\[ \hat{v}_1 := \text{TopEV}(C_y) \]

Centrality Estimation

Detected K possible central nodes
Blind centrality estimation

\[ C_y = \mathcal{H}(A)BB^\top (\mathcal{H}(A))^\top = V^TBB^\top V \begin{bmatrix} h(\lambda_1) \\ \vdots \\ h(\lambda_N) \end{bmatrix} V^T \]

(for ‘strong’ low pass filter) \( \approx \text{const} \cdot v_1v_1^T \)

Lemma

Suppose \( h(\lambda_1) > \max_{j=2,\ldots,n} h(\lambda_j) \). Then it holds that

\[ \|c_{eig} - \hat{v}_1\|_2 = O\left(\frac{\max_{j=2,\ldots,n} |h(\lambda_j)|}{|h(\lambda_1)|}\right) = O(\eta) \]

- Centrality estimation may be **inaccurate** for ‘weak’ low pass filter (i.e., \( \eta \approx 1 \)).
Boosted centrality estimation

- A simple modification to *strengthen* the low-pass filter.
- Let $\rho > 0$. Consider

$$\tilde{H}(A) := H(A) - \rho I, \quad \tilde{h}_\rho(\lambda) := h(\lambda) - \rho.$$ 

- Let $\mu := \frac{\max_{j=2,\ldots,n} |\lambda_j|}{|\lambda_1|}$

- **Observation**: there exists $\rho > 0$ such that

$$\frac{\max_{j=2,\ldots,n} |\tilde{h}_\rho(\lambda_j)|}{|\tilde{h}_\rho(\lambda_1)|} = O\left(\frac{\max_{j=2,\ldots,n} |\lambda_j|}{|\lambda_1|} \frac{\max_{j=2,\ldots,n} |h(\lambda_j)|}{|h(\lambda_1)|}\right) = O(\mu \eta)$$

- $\tilde{H}(A)$ has a **better low-pass condition** than $H(A)$. 
Boosted centrality estimation

- Assume the external signals \( Z \in \mathbb{R}^{k\times M} (k < M) \) are known, 
  \[
  \mathcal{H}(A)B =: \hat{HB} = YZ^\top(ZZ^\top)^{-1}
  \]

- \( \mathcal{H}(A)B \) admits a low-rank + sparse decomposition as:
  \[
  \mathcal{H}(A)B = \tilde{H}(A)B + \rho B \equiv L + S
  \]

- \( L \) is a low-rank matrix and \( S \) is a sparse matrix.

- To obtain \( L \), we solve the convex problem:
  \[
  \min_{\hat{L}, \hat{S}} \| \hat{HB} - \hat{L} - \hat{S} \|_F^2 + \lambda_L \| \hat{L} \|_* + \lambda_S \| \text{vec}(\hat{S}) \|_1
  \]

  where \( \lambda_L, \lambda_S \) – regularization for low-rankness, sparseness.
The whole process:

Corollary

Let $\tilde{v}_1$ be the top left singular vector of $L$. Under the same conditions as the previous Lemma. It holds

$$
||c_{eig} - \tilde{v}_1||_2 = O\left( \frac{\max_{j=2,\ldots,n} |\lambda_j|}{|\lambda_1|} \frac{\max_{j=2,\ldots,n} |h(\lambda_j)|}{|h(\lambda_1)|} \right) = O(\mu \eta)
$$
Numerical Results

- **Graph $G$:** Core periphery model with connectivity $p = 0.05$
  \[
  \begin{bmatrix}
  1 & 4p \\
  4p & p
  \end{bmatrix}
  \]

- **$N = 100$ nodes (10 core nodes), $M = 10^5$ observations**

- **Graph filter:** $\mathcal{H}(A) = (I - 0.1A)^{-1}$, $\lambda_L = 0.1$, $\lambda_S = 0.2 + \frac{2}{\sqrt{k}}$.

- **Three settings of $B$ for different locations of external sources** (black - central, blue - regular, red - external):

  Setting (a)/(b)/(c), $k = 20$

  Setting (b), $k = 50$
Numerical Results

\[ S_{\text{thres}} = 1(\hat{S} \geq 0.1) \odot \hat{S} \]
and replace \( \hat{L} \) with \( \hat{HB} - S_{\text{thres}} \).

Error rate = \( E\left[ \frac{1}{10} \left| \{1, \ldots, 10\} \cap \tilde{v}_1 \right| \right] \)

PCA suffers from a higher error rate than the robust methods.

The error rate for the robust methods decreases with \( k \).

Results are consistent with our theoretical analysis.
Real data

- Data: daily return data from S&P100 stocks in May 2018 to Aug 2019, consisting of $n = 99$ stocks and $m = 300$ samples, collected from https://alphavantage.co.
- External source: the latent input $z^t$ on the relevant days estimated from the interest level on Google Trend (https://trends.google.com) on $k = 5$ key words: ‘trade war’, ‘sales tax’, ‘Iran’, ‘oil crisis’ and ‘election’.
- Method: Robust Estimation with Quantization
Real data

- Estimated most influenced stocks:

**PCA:**
- NVIDIA
- NETFLIX
- Amazon

**Robust PCA:**
- GE
- ConocoPhillips
- Facebook

[Source: Wikipedia]
Real data

- Estimated most affected areas:

  - **Trade war**
    - Pharmaceutical industry (WBA, PEF, MDT)

  - **Sales tax**
    - Technology (INTC, ORCL, etc.)

  - **Oil crisis**
    - Oil field (e.g., SLB) and technology (e.g., QCOM, WBA)

  - **Election**
    - Technology (e.g., GE, EMR, etc.) and service (e.g., CVS, SBUX, COST) stocks

  - **Iran**
    - Food (KHC), finance (UNH, BLK), technology (LLY, ORCL), energy (EXC) and others (GM, HD).
Summary

- PCA works if the related filter is ‘strong’ low-pass.

- With ‘weak’ low-pass filter, boosting method is applied.

- Numerical experiments on synthetic and stocks data.

Future Question: Can we learn node centrality from data without knowing external sources?
References

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