Multiband TDOA Estimation from Sub-Nyquist Samples with Distributed Sensing Nodes

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Sensing scenario

- Wide frequency band of interest $W$
- $N$ communication sub-bands
  - a bandwidth of $B = W/N$
  - central frequency $f_{cn}$
- $K$ out of $N$ are occupied by
  
  $$x_k(t) = \bar{x}_k(t)e^{j2\pi f_k t},$$
  
  - $f_k \in \{f_{cn}^n\}_{n=1}^N$ - unknown
  - $\bar{x}_k(t)$ are w.s.s. and unknown
Introduction: multiband spectrum sensing

**Sensing scenario**

- Wide frequency band of interest $W$
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  - a bandwidth of $B = W/N$
  - central frequency $f^c_n$
- $K$ out of $N$ are occupied by
  
  \[ x_k(t) = \bar{x}_k(t) e^{j2\pi f_k t}, \]
  
  - $f_k \in \{f^c_n\}_{n=1}^N$ - unknown
  - $\bar{x}_k(t)$ are w.s.s. and unknown

**Sensing task**

- Acquire $s(t) = \sum_{k=1}^{K} x_k(t)$ at a sub-Nyquist rate (on the order of $KB$ rather than $W$)
- Find out which sub-bands are occupied, i.e., estimate $f_k$. 

\[ s(t) \]

\[ \text{Sensor} \]
Introduction: multiband sensing and TDOA estimation

- Multiple distributed time-synchronized wideband sensing nodes that exchange data with each other (or some centralized processing unit)
Introduction: multiband sensing and TDOA estimation

- **Multiple** distributed time-synchronized wideband sensing nodes that exchange data with each other (or some centralized processing unit)

- \( Q_{k,p} \leq Q \) multipath components in the propagation channel from \( k \)-th source to \( p \)-th sensor, each with an amplitude \( a_{k,p,q} \) and a delay \( \tau_{k,p,q} \)
Multiband sensing and TDOA Estimation: problem formulation

Source signals:

- to each source signal \( x_k(t) \) corresponds an autocorrelation function \( R_{kk}(\tau) \)

\[
R_{kk}(\tau) = \mathbb{E}\{x_k(t)x_k^*(t - \tau)\} = \bar{r}_{kk}(\tau)e^{j2\pi f_k \tau},
\]

where \( \bar{r}_{kk}(\tau) = \mathbb{E}\{\bar{x}_k(t)\bar{x}_k^*(t - \tau)\} \) is the baseband autocorrelation function.

- we assume that \( R_{k_1 k_2}(\tau) = \mathbb{E}\{x_{k_1}(t)x_{k_2}^*(t - \tau)\} \equiv 0 \ \forall \ k_1 \neq k_2. \)
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**Received signals:**

- The noise-free signal at the $p$-th sensor $s_p(t) = \sum_k \sum_{q=1}^{Q_{p,k}} a_{k,p,q} x_k(t-\tau_{k,p,q}) = \sum_k x_{k,p}(t)$
Multiband sensing and TDOA Estimation: problem formulation

Source signals:
- to each source signal $x_k(t)$ corresponds an autocorrelation function $R_{kk}(\tau)$
  \[ R_{kk}(\tau) = \mathbb{E}\{x_k(t)x^*_k(t-\tau)\} = \bar{r}_{kk}(\tau)e^{j2\pi f_k\tau}, \]
  where $\bar{r}_{kk}(\tau) = \mathbb{E}\{x_k(t)x^*_k(t-\tau)\}$ is the baseband autocorrelation function.
- we assume that $R_{k_1k_2}(\tau) = \mathbb{E}\{x_{k_1}(t)x^*_{k_2}(t-\tau)\} \equiv 0 \forall k_1 \neq k_2$.

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- for any pair of sensors $(p_1, p_2)$, we can calculate a cross-correlation function $R_{p_1,p_2}(\tau)$
  \[ R_{p_1,p_2}(\tau) = \mathbb{E}\{s_{p_1}(t)s^*_{p_2}(t-\tau)\} = \sum_k \sum_{q_1,q_2} \tilde{a}_{k,q_1,q_2}^{(p_1,p_2)} R_{kk}(\tau - \tilde{\tau}^{(p_1,p_2)}_{k,q_1,q_2}) = \sum_k \bar{r}_{k}^{(p_1,p_2)}(\tau)e^{j2\pi f_k\tau} \]
  where $\tilde{\tau}^{(p_1,p_2)}_{k,q_1,q_2}$ is the relative delay.
Multiband sensing and TDOA Estimation: problem formulation

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  \]

Sensing task:
- detect which sub-bands are active
- estimate relative autocorrelations $\tilde{r}_{k}^{(p_1,p_2)}(\tau)$ from sub-Nyquist samples of $s_p(t)$
Sub-Nyquist receiver system: the MWC

Modulated Wideband Converter:

- $M$ sampling channels
- analog compression in each channel
  - mixing with periodic sequences $p_m(t)$
  - low-pass filtering with cut-off $f_s/2$
- sampling at $f_s \rightarrow M$ low-rate digital outputs $y_m[n]$
- total sampling rate of $Mf_s$ vs. $W$

---

MWC operation in frequency domain

- analog mixing: \( p_m(t) \) is \( T_p \)-periodic \( \rightarrow \) it can be represented by Fourier series

\[
p_m(t) = \sum_{\ell=-\infty}^{\infty} c_{m,\ell} e^{j2\pi\ell f_p t}, \text{ where}
\]

\[
c_{m,\ell} = \frac{1}{T_p} \int_{T_p} p_m(t) e^{-j2\pi\ell f_p t} dt \quad \text{– weighted “Dirac-comb”}
\]
MWC operation in frequency domain

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- Multiplication in time $\leftrightarrow$ convolution in frequency:

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Introduction
Sub-Nyquist receiver system
Estimation methods
Numerical Example
Conclusions

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MWC operation in frequency domain

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\]

- multiplication in time ↔ convolution in frequency:

- low-pass filtering: only the narrowband part of the mixture around the origin is kept

- low-rate sampling with \( f_p = f_s \geq B \): each digital output contains all spectral parts of the original signal (rearranged and differently weighted)
MWC operation for time-delay estimation

- consider $m$-th digital output at $p$-th sensor\(^1\):

\[
\begin{align*}
\underbrace{y_{p,m}(f)}_{\text{DTFT of } y_{p,m}[nT_s]} &= [c_m, -L_0, \ldots, c_m, 0, \ldots c_m, L_0] \\
\begin{bmatrix}
S_p(f + L_0 f_p) \\
\vdots \\
S_p(f) \\
\vdots \\
S_p(f - L_0 f_p)
\end{bmatrix} &= c_m z_p(f)
\end{align*}
\]

- $y_{p,m}(f)$ is the DTFT of $m$-th discrete output of $p$-th sensor
- row-vector $c_m$ contains the Fourier coefficients of $p_m(t)$
- vector $z_p(f)$ contains $f_p$-shifted low-passed filtered copies of $s_p(f) = \int_{-\infty}^{\infty} s_p(t)e^{-2\pi fj}dt$

\(^1\)For simplicity, we assume that the sets of mixing functions at all sensing nodes are the same.
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\text{DTFT of } y_{p,m}[nT_s] &= c_m \begin{bmatrix} S_p(f + L_0f_p) \\ \vdots \\ S_p(f) \\ \vdots \\ S_p(f - L_0f_p) \end{bmatrix} = c_m z_p(f)
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- in time domain: $y_{p,m}[t_n = nT_s] = \frac{1}{T_s} \int_{T_s} e^{j2\pi fnT_s} df = c_m z_p[t_n]$

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\[ \text{IDTFT of } z_p(f) \]

- since \( f_k \in \{f_n^c\}_{n=1}^{N} \) and \( f_p = f_s \geq B \), only \( K \) entries of \( z_p[t_n] \) are non-zero

\[
y_{p,m}[t_n] = \sum_{k=1}^{K} c_{m,\ell_k} z_{p,\ell_k}[t_n] = \sum_{k=1}^{K} c_{m,\ell_k} \bar{x}_{k,p}[t_n] = \sum_{k=1}^{K} c_{m,\ell_k} \sum_{q=1}^{Q_{p,k}} a_{k,p,q} \bar{x}_{k}[t_n-\tau_{k,p,q}]
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- consider \( m \)-th digital output at \( p \)-th sensor:\

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\]

1 For simplicity, we assume that the sets of mixing functions at all sensing nodes are the same.
Estimation: joint recovery

Consider the cross-correlation between the $i$-th output of $p_1$-th sensor and the $j$-th output of $p_2$-th sensor:

$$r_{i,j}^{(p_1,p_2)}[\tau_\nu] = \mathbb{E}\{y_{p_1,i}[t_n]y_{p_2,j}[t_n-\tau_\nu]\} = \sum_{k=1}^{K} w_{i,j,\ell_k} \sum_{q_1,q_2} \tilde{a}_{k,q_1,q_2} r_{k,k}[\tau_\nu - \tilde{\tau}_{n,q_1,q_2}]$$

Concatenate all cross-correlations $r_{i,j}^{(p_1,p_2)}[\tau_\nu]$ together into one vector $r^{(p_1,p_2)}[\tau_\nu]$:

$r^{(p_1,p_2)}[\tau_\nu]$ is a $K$-sparse vector of length $L$ that contains unknown relative autocorrelation functions $\tilde{r}_{k}^{(p_1,p_2)}[\tau_\nu]$ at the positions with indices $\ell_k$ – support of $r_{i,j}^{(p_1,p_2)}[\tau_\nu]$ defines the central frequencies of the active sub-bands:

$$\forall \ell_k \in S \exists f_k = \ell_k f_p$$

$W$ is a matrix comprised of elements $w_{i,j,\ell_k}$ such that its $\ell_k$-th column is $w_{\ell_k} = [w_{1,1,\ell_k}, w_{1,2,\ell_k}, ..., w_{1,M,\ell_k}, w_{2,1,\ell_k}, ..., w_{M,M,\ell_k}]^T$.
Estimation: joint recovery

- consider the cross-correlation between the \(i\)-th output of \(p_1\)-th sensor and the \(j\)-th output of \(p_2\)-th sensor:

\[
r_{i,j}^{(p_1,p_2)}[\tau_{\nu}] = \mathbb{E}\{y_{p_1,i}[t_n]y_{p_2,j}^*[t_n-\tau_{\nu}]\} = \sum_{k=1}^{K} w_{i,j,\ell_k} \sum_{q_1,q_2} \tilde{a}_{k,q_1,q_2} \tilde{r}_{k,k}[\tau_{\nu} - \tilde{r}_{n,q_1,q_2}] \\
\]

- concatenate all cross-correlations \(r_{i,j}^{(p_1,p_2)}[\tau_{\nu}]\) together into one vector \(r_y^{(p_1,p_2)}[\tau_{\nu}]\):

\[
r_y^{(p_1,p_2)}[\tau_{\nu}] = W r_z^{(p_1,p_2)}[\tau_{\nu}] \\
\]
**Estimation: joint recovery**

- consider the cross-correlation between the $i$-th output of $p_1$-th sensor and the $j$-th output of $p_2$-th sensor:

  $$ r_{i,j}(p_1,p_2)[\tau_\nu] = \mathbb{E}\{y_{p_1,i}[t_n]y_{p_2,j}[t_n-\tau_\nu]\} = \sum_{k=1}^{K} w_{i,j,\ell_k} \sum_{q_1,q_2} \tilde{a}_{k,q_1,q_2}(p_1,p_2) \tilde{r}_{k_k}[\tau_\nu - \tilde{r}_{n,q_1,q_2}] $$

- concatenate all cross-correlations $r_{i,j}(p_1,p_2)[\tau_\nu]$ together into one vector $r_{y}(p_1,p_2)[\tau_\nu]$:

  $$ r_{y}(p_1,p_2)[\tau_\nu] = W r_{z}(p_1,p_2)[\tau_\nu] $$

  - $r_{z}(p_1,p_2)[\tau_\nu]$ is a $K$-sparse vector of length $L$ that contains unknown relative autocorrelation functions $\tilde{r}_{k_k}(p_1,p_2)[\tau_\nu]$ at the positions with indices $\ell_k$

  - support of $r_{z}(p_1,p_2)$ defines the central frequencies of the active sub-bands:

    $$ \forall \ell_k \in S \exists f_k = \ell_k f_p $$
Estimation: joint recovery

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concatenate all cross-correlations $r_{i,j}^{(p_1,p_2)}[\tau_{\nu}]$ together into one vector $r_{y}^{(p_1,p_2)}[\tau_{\nu}]$:

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- $r_{z}^{(p_1,p_2)}[\tau_{\nu}]$ is a $K$-sparse vector of length $L$ that contains unknown relative autocorrelation functions $\tilde{r}_{k}^{(p_1,p_2)}[\tau_{\nu}]$ at the positions with indices $\ell_k$
- support of $r_{z}^{(p_1,p_2)}$ defines the central frequencies of the active sub-bands: $\forall \ell_k \in S \exists f_k = \ell_k f_p$
- $W$ is a matrix comprised of elements $w_{i,j,\ell}$ such that its $\ell$-th column is

$$w_\ell = \begin{bmatrix} w_{1,1,\ell}, \ldots, w_{1,1,M}, w_{2,1,\ell}, \ldots, w_{2,1,M}, \ldots, w_{M,1,\ell}, \ldots, w_{M,M,\ell} \end{bmatrix}^T$$
Estimation: joint recovery

\[ r_{y}^{(p_1, p_2)}[\tau_{\nu}] = W r_{z}^{(p_1, p_2)}[\tau_{\nu}] \]

- typical sparse recovery problem → can be solved for each \( \tau_{\nu} \) independently
- we can apply the CTF block from (2)
Estimation: joint recovery

\[ r^{(p_1,p_2)}_y[\tau_\nu] = W r^{(p_1,p_2)}_z[\tau_\nu] \]

- typical sparse recovery problem → can be solved for each \( \tau_\nu \) independently
- we can apply the CTF block from (2)

Once the support \( S \) of \( r^{(p_1,p_2)}_z[\tau_\nu] \) is found we obtain

- the central frequencies of the active sub-bands
  \[ f_k = \ell_k f_p, \ \ell_k \in S \]
- the discrete baseband relative autocorrelation functions \( \tilde{r}_k^{(p_1,p_2)}[\tau_\nu] \) as
  \[
  \left( r^{(p_1,p_2)}_z \right)_S[\tau_\nu] = W^\dagger_S r^{(p_1,p_2)}_y[\tau_\nu],
  \]
  where for some vector \( a \) and matrix \( A \) the notation \( a_S \) and \( A_S \) means taking the entries of \( a \) and the columns of \( A \) indexed by \( S \), respectively.

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Estimation: two-step recovery

Alternatively, we can

- first estimate the support $S$ and the corresponding low-rate sequences $\bar{x}_{k,p}[t_n]$ from the outputs of individual sensors
- compute $\bar{r}_{k}^{(p_1,p_2)}[\tau_\nu]$ for each $k$ independently
Estimation: two-step recovery

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Estimation procedure:

- collect all $M$ outputs $y_{p,m}[t_n]$ of the $p$-th sensor together into one vector $y_p[t_n]$
  
  $$y_p[t_n] = [c_1, \cdots, c_M]^T z_p[t_n]$$

- find the support of $z_p[t_n]$ from $y_p[t_n]$
  - the support of $z_p[t_n]$ is also $S$
  - as before, we can find it either for each $t_n$ or by applying the CTF block

- estimate the individual low-rate sequences $\bar{x}_{k,p}[t_n]$ via $(z_p)_S[t_n] = C^+_S y_p[t_n]$

- obtain baseband relative autocorrelations $\bar{r}_{k}^{(p_1,p_2)}[\tau_\nu]$
  
  $$\bar{r}_{k}^{(p_1,p_2)}[\tau_\nu] = \mathbb{E}\{\bar{x}_{k,p_1}[t_n] \bar{x}_{k,p_2}[t_n - \tau_\nu]\}$$
Numerical example: simulation setup

Sensing scenario:
- frequency band of $W = 3.9$ GHz is split into $N = 135$ communication channels
- $W$ is occupied by $K = 3$ BPSK modulated signals $x_k(t)$
  - bandwidth of $B = 20$ MHz
  - carrier $f_k$ chosen uniformly at random from $\{f_c^n\}_{n=1}^N$

Propagation parameters:
- number of multipath components is $Q_{k,p} = 2$
- time delays $\tau_{k,p,q}$ and amplitudes $a_{k,p,q}$ are chosen uniformly at random from $[\frac{N_T}{100}, \frac{9N_T}{100}]$ and $[0.6, 1]$
  - $N_T$ is the sensing time in samples

Sensors:
- sensors operate with $M = 20$ sampling channels
- the sampling rate is $f_s = 28$ MHz
- $p_m(t)$ are generated as pseudo-random $\{\pm 1\}$ piece-wise constant functions
- total sampling rate at each sensor is $560$ MHz, which is $14\%$ of the Nyquist rate

Performance metrics:
- support recover rate (SRR): $|\hat{S} \cap S| / K$
- mean square error (MSE) between the true and the estimated relative autocorrelation functions:
  $$\frac{1}{K N_T} \sum_{k=1}^K \sum_{\nu=1}^{N_T} \frac{|\hat{r}_k(p_1,p_2)[\tau]\nu| - \hat{r}_k(p_1,p_2)[\tau]\nu|^2}{|\hat{r}_k(p_1,p_2)[\tau]\nu|^2}$$
Numerical example: performance vs SNR

\( N_T = 500 \) samples

- Two-step recovery approach provides somewhat higher SRR rate
- Joint estimation method provides slightly better accuracy in terms of \( \bar{r}_k^{(P_1;P_2)}[\tau_\nu] \) recovery
**Numerical example: performance vs sensing time**

**SNR = 0 dB**

The same tendency

Two-step approach is less sensitive to sensing time duration for support recovery
Conclusions

- we considered the task of relative autocorrelation estimation of multiple unknown transmitters from the sub-Nyquist samples of wideband multiband signals obtained by a network of spatially distributed sensing nodes.

- we showed that the central frequencies and the relative autocorrelation functions of the individual transmissions can be estimated from the low-rate outputs of different sensors and proposed two estimation methods
  - joint recovery of the frequency support and the relative autocorrelation functions
  - two-step approach

- both proposed methods allow for central frequency and relative autocorrelation estimation from sub-Nyquist samples
  - the joint recovery yields an improved accuracy of the latter while being more sensitive with respect to the sensing time
Thank you! Questions?