One-Bit Unlimited Sampling
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Introduction & Contribution

Key Takeaways
- Unlike Shannon’s Sampling Theorem, the analog-to-digital converters (ADCs) are limited in dynamic range, thus prone to saturation and clipping. In order to circumvent these problems, the authors introduced the concept of Unlimited Sampling in [1].
- Behind work [1] are recent developments in ADC design - the Self-Reset ADCs, which compute modulo samples [2].
- The Unlimited Sampling Theorem proves that a bandlimited signal can be perfectly recovered from modulo samples. The sampling rate is independent of the ADC threshold.
- As a step towards practical implementation, we consider not only sampling, but also quantization.
- We combine the advantages of Unlimited Sampling and One-Bit Sigma-Delta Quantization (SDQ) to obtain an ADC scheme that has low complexity due to coarseness of quantization and at the same time overcomes the dynamic range limitations of conventional One-Bit SDQ.

Unlimited Sampling of Bandlimited Functions

- Let \( \tau \geq 1 \) be the (over)sampling rate and \( g(t) \) be a \( \tau \)-bandlimited function.
- In Unlimited Sampling framework, we sample \( g \) using non-linear principle:
  \[
  y[n] = \text{mod}_\lambda (g(2^n)), \quad n \in \mathbb{Z}, \quad \tau \geq 1
  \]
- Such folded samples are acquired using a version of the Self-Reset ADC [2].
- Even if \( g(t) \geq \lambda \), \( y[n] \in [0, \lambda) \). In this work, we set \( \lambda = 1 \).

Unlimited Sampling in Action

- A sufficient condition for recovery of \( \tau \)-bandlimited signal \( g \) from its modulo samples \( y[n] = \text{mod}_\lambda (g(2^n)) \) up to additive multiples of \( 2\lambda \) is \( \tau > \pi e \).

Unlimited Sampling Meets One-Bit Quantization

- In order to discretize the range, \( y[n] \) is quantized via the first order, one-bit Sigma-Delta Quantizer:
  \[
  u[n] = u[n-1] + y[n] - q[n], \quad q[n] = \text{sign}(u[n-1] + y[n])
  \]
- System architecture for One-Bit Unlimited Sampling:

Conventional One-Bit Sampling vs One-Bit Unlimited Sampling

- Conventional one-bit samples exhibit saturation if the dynamic range exceeds \([-1,1]\).
- The self-reset ADC folds modulo samples. In contrast, the self-reset ADC is able to fold modulo samples over a finite range.

Recovery from One-Bit Modulo Samples


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Sufficiency Condition and Error Bound

One-Bit Unlimited Sampling Theorem

Given
- \( g \in B_\lambda \) and not superoscillating, \( \beta \geq \|g\|_\infty \).
- \( q[n] \) - the one-bit modulo samples of \( g(t) \).
- \( \psi_\lambda^n(t) \) - the samples of the smoothing kernel \( \psi_\lambda(t) \) with sampling rate \( h_k := 2 \|\partial^2 \psi_\lambda^n(t)\|_1 \),
- a valid reconstruction kernel \( \varphi(t) \),

a sufficient condition for approximate recovery of \( \tilde{g}(t) \) from \( q[n] \) (up to additive multiples of 2) is

\[
\tau > 4\pi e \lambda \left( \|\partial^2 \psi_\lambda^n\|_1 \|\psi_\lambda^n\|_1 + 1 \right).
\]

Under these conditions, Recovery Algorithm yields the reconstruction error

\[
\|g(t) - \tilde{g}(t)\|_1 \leq \frac{1}{\tau} \|\partial^2 \varphi\|_1 + M(\varphi, \psi_\lambda^n),
\]

where \( M(\varphi, \psi_\lambda^n) \) is a constant dependent on the choice of kernels \( \varphi \) and \( \psi_\lambda^n \).

Our algorithm allows for recovery with accuracy \( O(1/\tau) \), which is close to the best known error bound \( O(1/\tau^{3/2}) \) for conventional first order SDQ [3].

Reconstruction Example

- Randomly generated \( \tau \)-bandlimited signal \( g \), its one-bit modulo samples \( q[n] \) acquired with \( \tau = 250 \) and the reconstructed signal \( \tilde{g} \) which is obtained using second order \( \psi_\lambda^n \). The mean error \( \|g - \tilde{g}\|_1 = 2.1 \times 10^{-5} \).

- The true residual \( \varepsilon_\lambda \) and its approximate recovery \( \tilde{\varepsilon}_\lambda \).

References