Wirtinger Flow Method with Optimal Stepsize for Phase Retrieval

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Abstract

The recently reported Wirtinger flow (WF) algorithm has been demonstrated as a promising method for solving the problem of phase retrieval by applying a gradient descent scheme. An empirical choice of stepsize is suggested in practice. However, this heuristic stepsize selection rule is not optimal. In order to accelerate the convergence rate, we propose an improved WF with optimal stepsize. It is revealed that this optimal stepsize is the solution of a univariate cubic equation with real-valued coefficients.

Finding its roots is computationally simple because a closed-form expression exists.

Introduction

Phase retrieval refers to the task of recovering a signal from phaseless measurements, i.e., reconstruction of a signal given only the magnitudes of its linear measurements. This problem arises in various fields of science and engineering, such as optical imaging, X-ray crystallography, astronomy, and radar, when the phases of the linear transform of the signal are unavailable [1].

Wirtinger Flow with Optimal Stepsize (WFOS)

Given \( x^0 \) and \( g^0 \) at the 0th iteration, the optimal step size can be obtained by solving the line search

\[
\beta^* = \arg \min \{ f(x^0 - \alpha g^0) \}
\]

Note that the cost function \( f(x^0 - \alpha g^0) \) is a univariate quartic function of \( \alpha \). The optimal stepsize satisfies the following first-order optimality condition:

\[
\frac{d}{d\alpha} f(x^0 - \alpha g^0) = 0
\]

which leads to a univariate cubic equation of \( \alpha \) given by

\[
\epsilon_1 \alpha^3 + \epsilon_2 \alpha^2 + \epsilon_3 \alpha + \epsilon_4 = 0
\]

with real-valued constant coefficients \( \{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\} \). Then the optimal stepsize is the real root associated with the minimum objective value if it has three real roots or the minimizer is the unique real root if it has a real root and a pair of complex conjugate roots.

Computational Complexity:

- The computational cost for calculating the coefficients \( \{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\} \) is \( \mathcal{O}(M) \).
- The complexity of calculating the roots of a cubic equation is merely \( \mathcal{O}(1) \) because a closed-form solution exists.
- Computing the gradient is still the leading cost of the proposed WFOS in each iteration, which is \( \mathcal{O}(MN) \).

Simulation Results

Simulation settings:

- Random complex Gaussian signal
- Signal dimension \( N \approx 64 \); Observation dimension is six times the signal dimension, i.e., \( M = 6N \).
- The same initial value obtained from the spectral method [1].
- The stepsize for WF at the 0th iteration is \( \alpha_0 = \min\{1 - \frac{1}{n+1}a, 0.2\} \).
- The noise component is sampled from \( \mathcal{CN}(a^2/2, N(0,a^2)/2) \) and we have \( SNR = 20 \text{ dB} \).

Fig. 1: The convergence behavior of the WFOS compared with WF. The light curves represent the results for each run and thick curves represent the average results over 50 Monte Carlo trials.

Conclusion

- The proposed WFOS for phase retrieval significantly accelerates the convergence rate of WF.
- The optimal stepsize is demonstrated to be the solution of a univariate cubic equation with real-valued coefficients.
- The WFOS has the same leading cost as WF for computation of the gradient in each iteration, which is \( \mathcal{O}(MN) \).
- The proposed scheme to obtain the optimal stepsize of WF can also be directly applied to the truncated WF.

References