# Small Perturbation Analysis of Network Topologies

*Elena Ceci and Sergio Barbarossa*

*Dept. of Information Engineering, Electronics, and Telecommunications,*
*Sapienza University of Rome, Via Eudossiana,18, 00184, Rome, Italy*

# Motivations

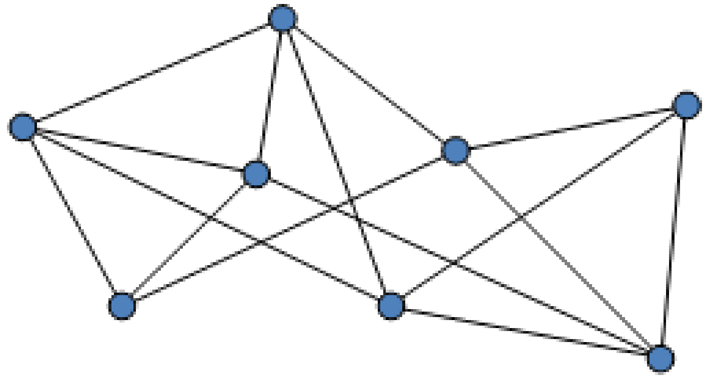
- Analysis of network macroscopic changes induced by random perturbation of a limited percentage of edges
- Analysis of random edge perturbations in graph-based learning methods (clustering)
- Robust signal processing over uncertain graphs
- Robust resource allocation over networks

# Graph representation

**Definitions**:

For a given graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$

- Matrix of similarity/adjacency $\boldsymbol{A} = \begin{cases} a_{i,j} > 0, & (v_i, v_j) \in \mathcal{E} \\ 0 & otherwise \end{cases}$

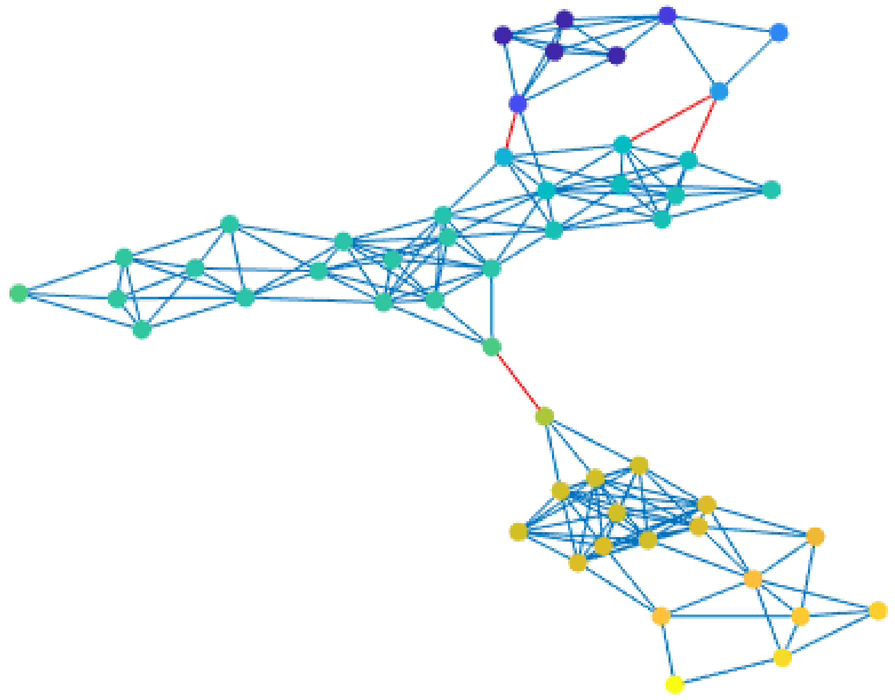


- Laplacian $\boldsymbol{L} = \boldsymbol{D} - \boldsymbol{A}$ $where$ $\boldsymbol{D} = diag(\mathbf{1}^T \boldsymbol{A})$

Eigen-decomposition of Laplacian $\boldsymbol{L} = \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^T = \sum_{i=1}^{N} \lambda_i \boldsymbol{u}_i \boldsymbol{u}_i^T$

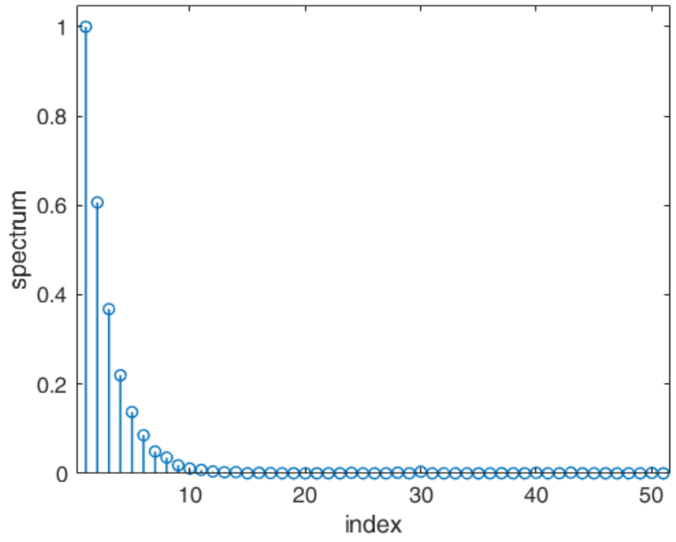
# Robust signal processing on uncertain graph

Example of perturbed graph

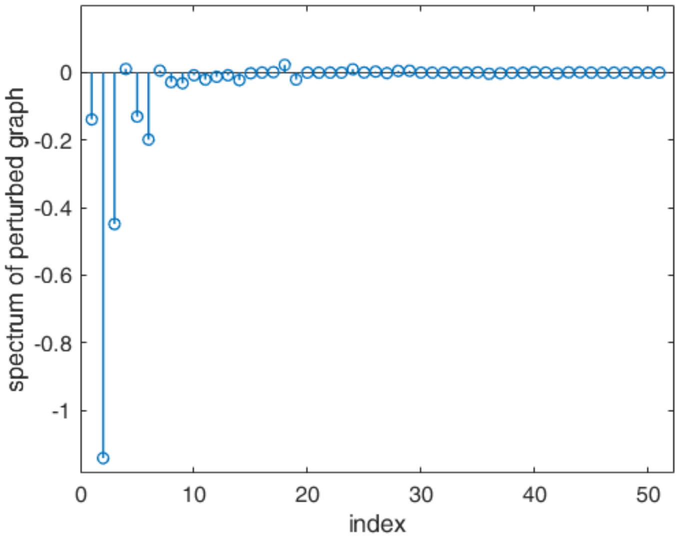


Smooth signal over vertices of a graph and deleted edges (in red)

Smooth signal spectrum




Effect of deletions on the spectrum



# Spectral clustering

Finding partitions $A_i, i = 1, \dots, K$ of a graph is equivalent to minimize RatioCut function [1]

$$RatioCut(A_1, \dots, A_K) \coloneqq \sum_{i=1}^{K} \frac{cut(A_i, \overline{A_i})}{|A_i|}$$

if $[\boldsymbol{H}]_{i,j} = \begin{cases} 1/\sqrt{|A_j|}, & if\ v_i \in A_j \\ 0, & otherwise \end{cases}$ and $\boldsymbol{h}_i, i = 1, \dots, N$ orthogonal columns of $\boldsymbol{H}$

$$RatioCut(A_1, \dots, A_K) = \sum_{i=1}^{K} \boldsymbol{h}_i^T \boldsymbol{L} \boldsymbol{h}_i = Tr(\boldsymbol{H}^T \boldsymbol{L} \boldsymbol{H}),$$

Problem statement

$$\min_{A_1, \dots, A_K} Tr(\boldsymbol{H}^T \boldsymbol{L} \boldsymbol{H}) \ s.t. \boldsymbol{H}^T \boldsymbol{H} = \boldsymbol{I}$$

➡ NP-hard

[1] U. von Luxburg, “A tutorial on spectral clustering,” Statistics and Computing, vol. 17, no. 4, pp. 395–416, Dec 2007

# Problem relaxation: $\boldsymbol{H} \in \mathbb{R}^{N \times K}$

Solution of relaxed problem [1]:

- take the K eigenvectors associated to the first $K$ eigenvalues $\boldsymbol{H} = \boldsymbol{U}_K$ and

$$Tr(\boldsymbol{H}^T \boldsymbol{L} \boldsymbol{H}) = \sum_{i=1}^{K} \lambda_i$$

- apply *k-means* (or equivalent method) to cluster

The first $K$ eigenvector-eigenvalue pairs contain the most important information about clustering properties



[1] U. von Luxburg, "A tutorial on spectral clustering," Statistics and Computing, vol. 17, no. 4, pp. 395–416, Dec 2007

# Small Perturbation Theory

Let $\mathcal{E}$ the set of existing links and $\bar{\mathcal{E}}$ its complement set and $m = (i_m, f_m) \in \mathcal{E} \cup \bar{\mathcal{E}}$

$$\delta L(m) = \underset{\underset{m \in \mathcal{E}}{\downarrow}}{\overset{\overset{m \in \bar{\mathcal{E}}}{\uparrow}}{+/-}} \quad \begin{array}{cc} & \begin{array}{ccccc} \phantom{0} & i_m & \phantom{\cdots} & f_m & \phantom{0} \end{array} \\ \begin{array}{c} \\ i_m \\ \\ f_m \\ \\ \end{array} & \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ \vdots & 1 & \vdots & -1 & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & -1 & \vdots & 1 & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}, \end{array}$$

Laplacian perturbation matrix of $m$-th edge

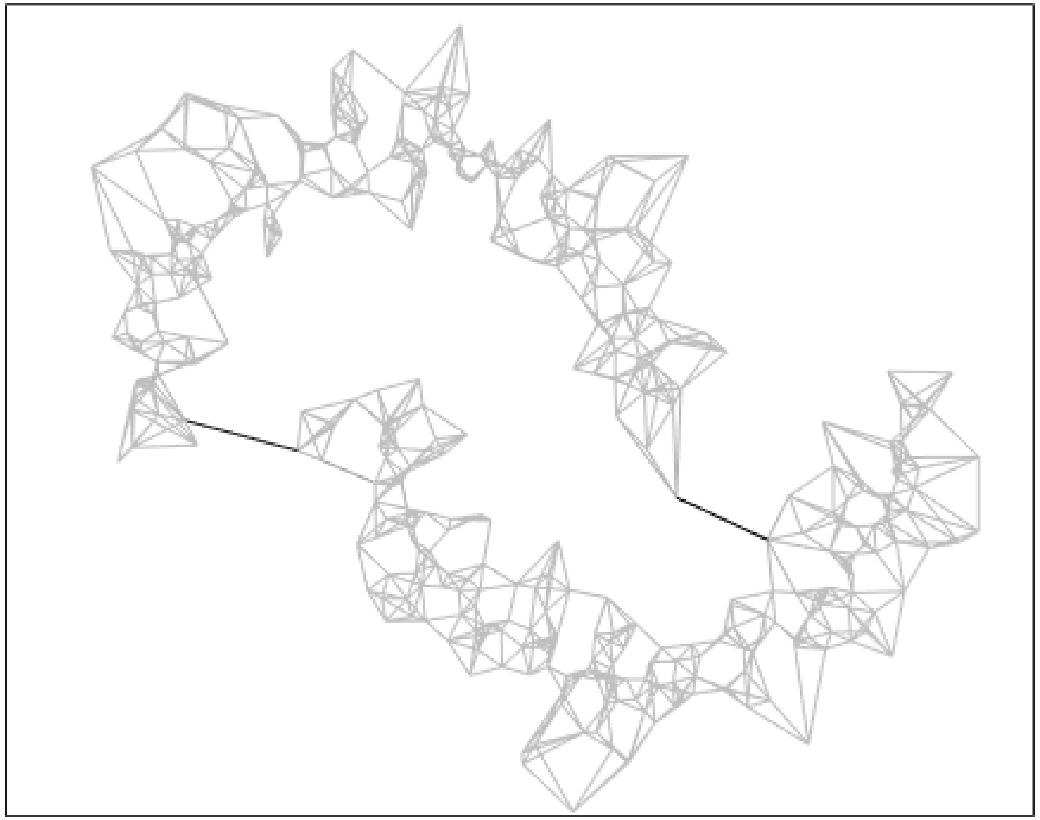
parallel perturbations $\delta L = \sum_{m \in \mathcal{E} \cup \bar{\mathcal{E}}} \delta L(m)$

perturbation of Laplaican eigenvalues/eigenvectors of multiplicity 1 within the first order of approximation in case of appearance/deletion of $m$-th edge :

$$\Delta\lambda_i(m) \simeq \boldsymbol{u}_i{}^T \delta L(m) \boldsymbol{u}_i = \pm\big(u_i(f_m) - u_i(i_m)\big)^2$$

$$\Delta\boldsymbol{u}_i(m) \simeq \sum_{j \neq i} \frac{\boldsymbol{u}_j^T \delta L(m) \boldsymbol{u}_i}{\lambda_i - \lambda_j} \boldsymbol{u}_{\boldsymbol{j}} = \pm \sum_{j \neq i} \frac{(u_j(i_m) - u_j(f_m))(u_i(i_m) - u_i(f_m))}{\lambda_i - \lambda_j} \boldsymbol{u}_{\boldsymbol{j}}$$

# New centrality measure



The color of each edge encodes $p_2(m)$

Definition of topology perturbation centrality

$$\boxed{\mathrm{p}_K(m) = \sum_{k=2}^{K} |\Delta\lambda_k|}$$

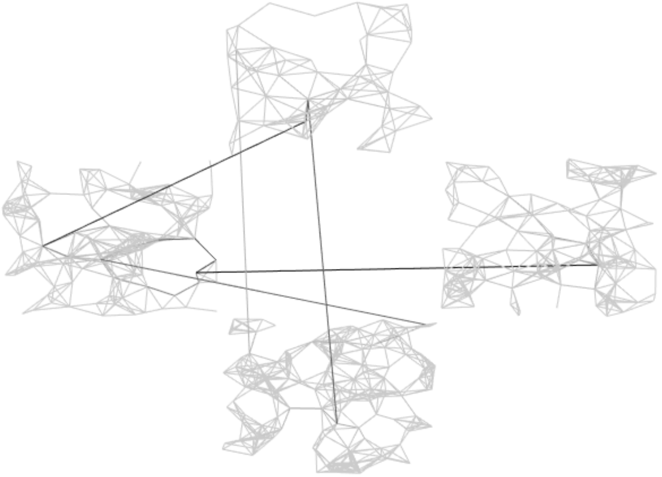
Where $K$ is the number of clusters

The perturbation measure associates a positive weight to each edge measuring how much the deletion of that edge modifies the modularity structure of the network

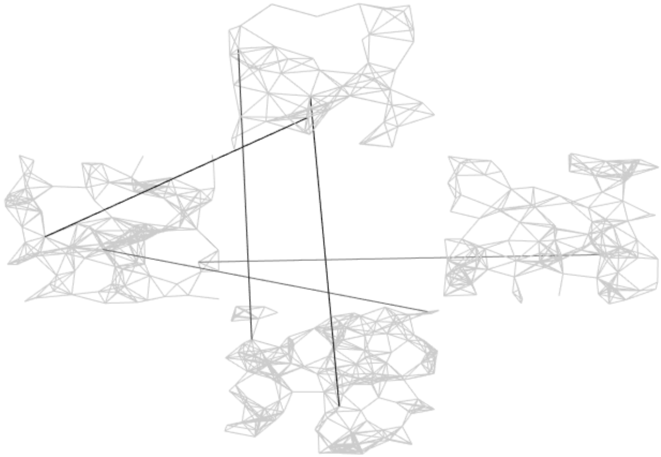
# New centrality measure and Betweenness

## Similarity between Perturbation topology centrality and flow betweenness

**Betweenness $\boldsymbol{B}$**

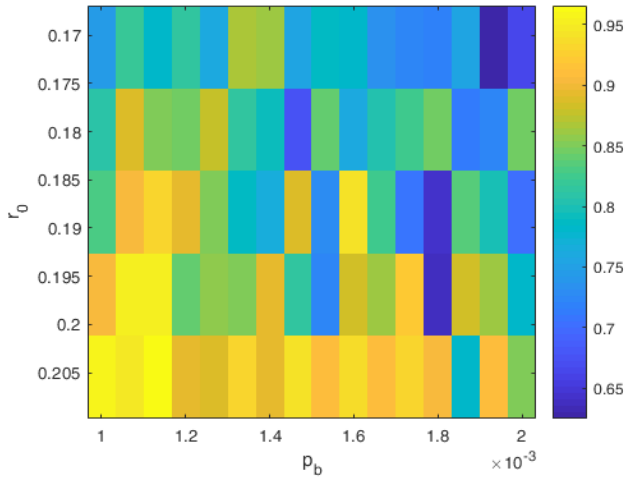


**Perturbation centrality $\boldsymbol{p_K}$**



Correlation between the two measures, where $M_L$ is the number of the most important links

$$\rho = \frac{\sum_{l=1}^{M_L} B(l)\mathrm{p}_K(l)}{\sqrt{\sum_{l=1}^{M_L} B^2(l)\ \sum_{l=1}^{M_L} \mathrm{p}_K^2(l)}}$$




$r_0$ coverage radius of each RGG
$p_b$ the probability of having bridge edges

The two different measures identify the same links as the important ones for modular graphs

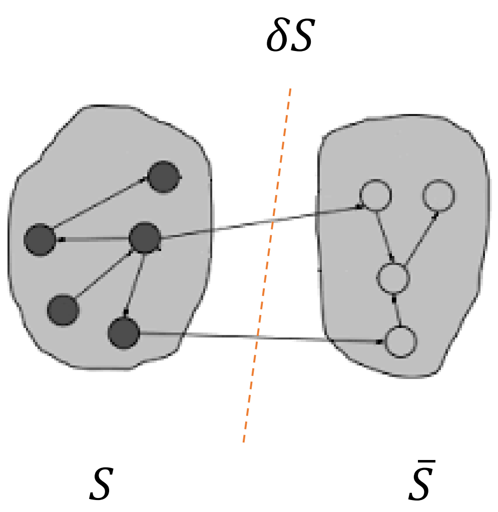
# Robust information transmission over random networks

Max flow/min cut theorem :

The maximum flow between a given pair of vertices in a network is equal to the sum of the weights on the edges of the minimum edge cut set that separates the same two vertices.

Conductance $\Phi(G) := \min_{S \subseteq V; 1 \le |S| \le N/2} \frac{|\delta S|}{|S|} \ge \frac{\lambda_2}{2}$

Where $\delta S = \{(\mathrm{i}, \mathrm{j}) \in E : \mathrm{i} \in S \text{ and } \mathrm{j} \in \bar{S}\}$ is the boundary of the vertices set S




$\lambda_2$ can be used as a parameter to assess the network **connectivity**

$\Delta\lambda_2$ can be used as a parameter to assess the **perturbation** of network **connectivity**

# Robust resource allocation

Goal: Minimize the average perturbation of algebraic connectivity $\mathrm{E}\{|\Delta\lambda_2|\}$

Cost: Sum of transmit power over each link $P_T = \sum_{m\in E} P_T(m)$

where $P_T(m)$, for each edge $m$, is a function of the outage probability $P_{out}(m)$:

$$P_T(m) = \frac{\sigma_n^2 r_m^2 (2^R - 1)}{F_n^{-1}(P_{out}(m); \lambda)}$$

$\sigma_n^2$ noise variance
$r_m$ distance covered by link $m$
$R$ data rate

with $F_n^{-1}(P_{out}(m); \lambda)$ is the inverse of Gamma CDF

Problem formulation:

$$\min_{P_{out}(m)\in[0,1]} \sum_{m\in\mathcal{E}} P_{out}(m)|\Delta\lambda_2(m)|$$

$$s.t. \quad \sum_{m\in\mathcal{E}} P_T(m) \le P_{T_{max}}$$

Change of variable:

$t_m = 1/F_n^{-1}(P_{out}(m);\lambda)$

Generalized convex problem in a bounded region:

$$\min_{\boldsymbol{t}} \sum_{m\in\mathcal{E}} F_n(\frac{1}{t_m};\lambda)|\Delta\lambda_2(m)|$$

$$s.t. \quad \sum_{m\in\mathcal{E}} r_m^2 t_m \le C_{max} = \frac{P_{T_{max}}}{\sigma_n^2(2^R-1)}$$

$$t_m \ge \frac{\lambda}{(n+1)}, m \in \mathcal{E}$$

**NON CONVEX**

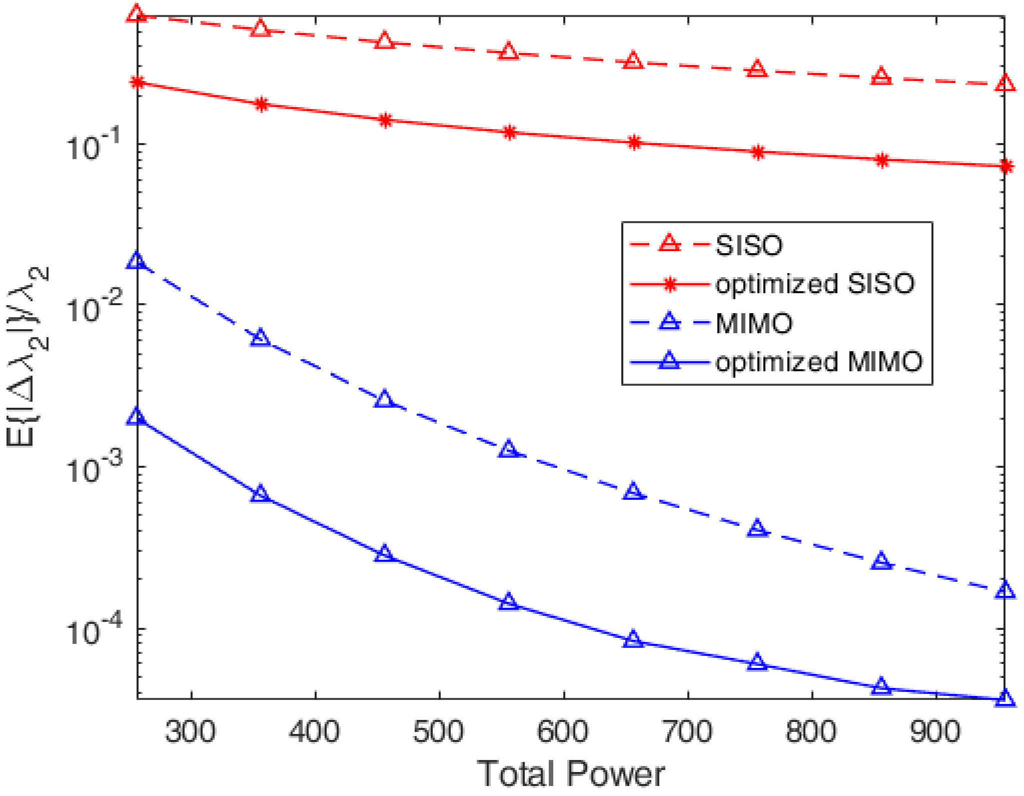
**CONVEX**

with $F_n\left(\frac{1}{t_m};\lambda\right)$ the CDF of Gamma distribution in MIMO system case, and $n$ the number of independent channels.

## SISO vs. MIMO

- The overall power needed to ensure the same perturbation, in the SISO case, is about 10 times smaller using the proposed optimization
- MIMO systems find better solutions exploiting the diversity gain
- The optimization gain is obtained allocating more resources to the most critical links

K=2 clusters network with four bridge edges




In blue the case with n=4 and in red the case with n=1

Overall network Total Power= $\sum_{m\in\mathcal{E}} P_{T_{max}}(m), m = 1, \ldots, 1612$

# Conclusions

- Small perturbation analysis is useful to find closed form and insightful expressions, albeit approximated, of perturbed eigenvalues and eigenvectors of a stochastic graph Laplacian

- We introduced a new centrality measure based on approximated closed forms of eigenvalues perturbation

- The statistical analysis of small perturbation has been used to find out a robust resource allocation over random networks

- Further developments :

    - robust clustering
    - robust analysis of signals defined over a graph, under imperfect knowledge of graph topology
    - tracking signals over uncertain graphs