**FAST SAMPLING OF GRAPH SIGNALS WITH NOISE VIA NEUMANN SERIES CONVERSION**

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**GRAPH SIGNAL PROCESSING**

- Signals on irregular data kernels: $q = (V, W)$
  - Combinatorial Laplacian matrix $L = D - W$ where degree matrix $D$ is a diagonal matrix with node degrees $d_i = \sum_j w_{ij}$
  - Graph Fourier transform (GFT): $L = V\Sigma V^T$
  - Bandlimited graph signal $x = \hat{x} V$ where the GFT coefficient $\hat{x}$ are non-zero only at the first $K$ elements
  - Image patch on 2D grid: $x = \hat{x} V$

**SAMPLING OF NOISY BANDLIMITED GRAPH SIGNAL**

- Motivation: sensing (acquiring samples) is expensive.
- Goal: sampling the most informative nodes for signal reconstruction.
- Signal model: noisy bandlimited graph signal $x = \hat{x} V + \tilde{x}$

Previous works:

- Aggressive sampling
- Random sampling
- Local reconstruction
- Residual selection

**APPLICATIONS**

- Sensor selection
- Active learning
- Matrix completion

**AUGMENTED A-OPTIMAL GRAPH SAMPLING**

- A-optimal sampling based on least square reconstruction: $\hat{x} = CV_x$ where $C$ is a sampling operator corresponding to $G$
  - Noisy observation: $y = \hat{x} + n$
  - Matrix completion: $\hat{x} = \hat{x} V (CV_x)^{-1} y$
  - Reconstruction MSE: $\mathbb{E}[(\hat{x} - \hat{x})^2]$

**RECOMMENDED OPTIMALITY CRITERION**

- Perfect reconstruction: $\mathbb{E}[(\hat{x} - \hat{x})^2] = 0$
- Minimal variance unbiased reconstruction: $\mathbb{E}[(\hat{x} - \hat{x})^2] = \text{Var}(x) / \text{SNR}$

**Augmented A-optimal sampling criterion**

- $C = \arg \min \mathbb{E}[(\hat{x} - \hat{x})^2]$

**NOISY NEUMANN SERIES THEOREM**

If the absolute value of eigenvalues of $A$ are all in the range $[\lambda-1, \lambda+1]$, then its Neumann Series converges: $\sum_{i=0}^{\infty} A^i = \frac{1}{\lambda I - A}$

**PROPOSED OBJECTIVE FUNCTION**

- Minimize $\mathbb{E}[(\hat{x} - \hat{x})^2]$

**SHIFT PARAMETER DESIGN**

- Design of $\mu$ based on inverse computation stability
- Inverse of matrix $G_{\mu}$ unstable if $\mu$ is extremely small since its eigenvalues are in $[\mu, 1 + \mu]$.
- We propose to bound the condition number of $G_{\mu}$

**RECOMMENDED MSE of different $\mu$ at 0dB**

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**EXPERIMENTAL RESULTS**

- In experiments, we set $K_0 = 100$. Reconstruction MSE is not sensitive to the choice of $\mu$ in community graph at 0dB.

**REFERENCES**