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Lecture Session SPTM-L3: Convex Optimization

Performance Analysis of Discrete-Valued Vector Reconstruction Based on Box-Constrained Sum of L1 Regularizers

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Outline

1. Introduction
2. Main Result
3. Simulation Results
4. Conclusion

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1. **Introduction**
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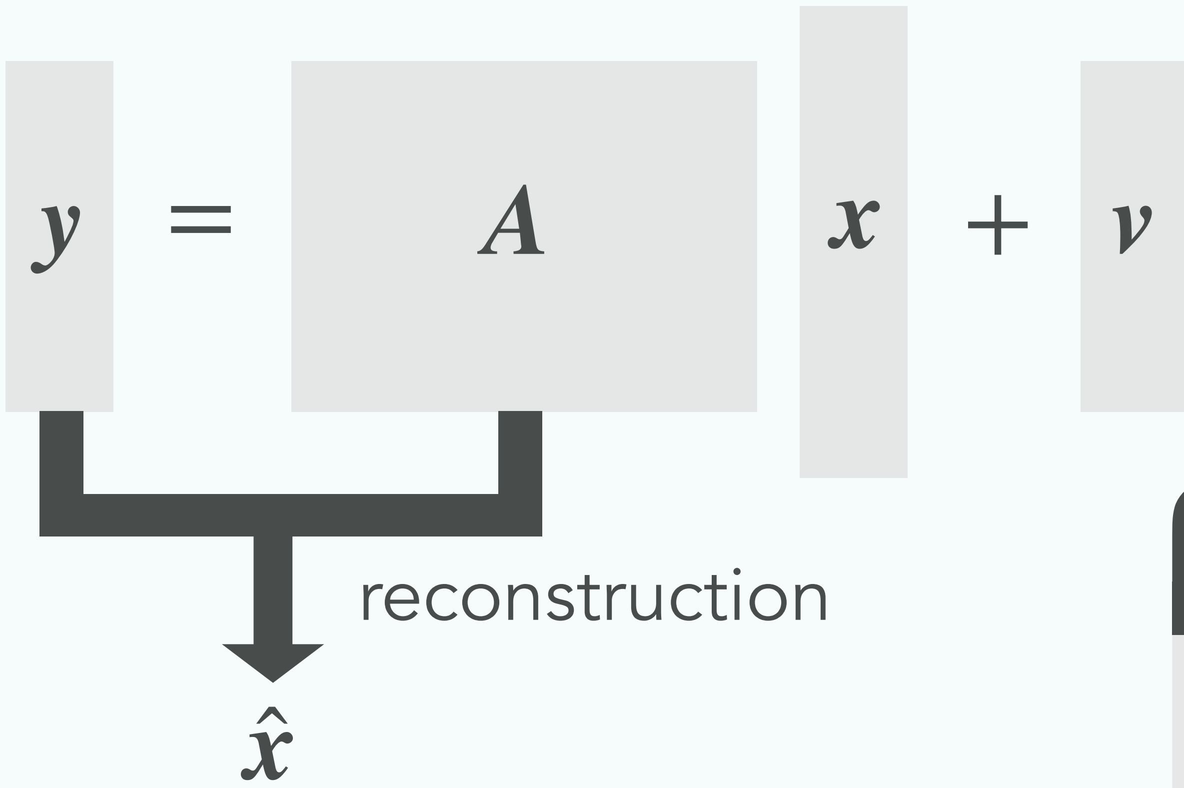
Discrete-Valued Vector Reconstruction

Reconstruction of a **discrete-valued** vector $\boldsymbol{x} \in \{r_1, \ldots, r_L\}^N$ $(r_1 < \cdots < r_L)$ from its **underdetermined** linear measurement $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{v} \in \mathbb{R}^M$ $(M < N)$



Application

- overloaded MIMO signal detection [1]
- multiuser detection [2]
- faster-than-Nyquist signaling [3]

[1] K. K. Wong, A. Paulraj, and R. D. Murch, “Efficient high-performance decoding for overloaded MIMO antenna systems,” IEEE Trans. Wireless Commun., vol. 6, no. 5, pp. 1833–1843, May 2007.

[2] H. Zhu and G. B. Giannakis, ‘’Exploiting sparse user activity in multiuser detection,” IEEE Trans. Commun., vol. 59, no. 2, pp. 454–465, Feb. 2011.

[3] J. E. Mazo, “Faster-than-Nyquist signaling,” Bell Syst. Tech. J., vol. 54, no. 8, pp. 1451–1462, 1975.

Reconstruction Methods (1/3)

✦ Linear minimum mean-square-error (LMMSE) approach

✓ Low complexity
- The performance is degraded in **underdetermined** problems

✦ Maximum likelihood (ML) approach

perform exhaustive search to reconstruct $\boldsymbol{x} \in \{r_1, \ldots, r_L\}^N$

$$\underset{\boldsymbol{s} \in \{r_1, \ldots, r_L\}^N}{\text{minimize}} \quad \frac{1}{2}\|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{s}\|_2^2$$

✓ Good performance
- The computational complexity is prohibitive in **large-scale** problems

In **large-scale underdetermined** problems, we require a low-complexity method which can achieve reasonable performance

Reconstruction Methods (2/3)

✦ Box relaxation method [4]

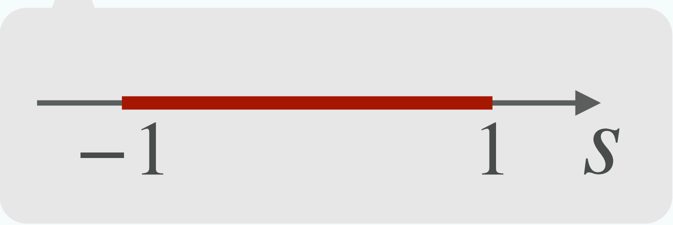
relax the ML method to convex optimization
under the box constraint $\boldsymbol{s} \in [r_1, r_L]^N \quad (r_1 < \cdots < r_L)$

$$\underset{\boldsymbol{s} \in [r_1, r_L]^N}{\text{minimize}} \; \frac{1}{2} \|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{s}\|_2^2$$

(binary phase shift keying)

Ex. For the reconstruction of BPSK signals $\boldsymbol{x} \in \{-1, 1\}^N$,
box relaxation problem is given by

$$\underset{\boldsymbol{s} \in [-1, 1]^N}{\text{minimize}} \; \frac{1}{2} \|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{s}\|_2^2$$



[4] P. H. Tan, L. K. Rasmussen, and T. J. Lim, "Constrained maximum-likelihood detection in CDMA," IEEE Trans. Commun., vol. 49, no. 1, pp. 142–153, Jan. 2001.

Reconstruction Methods (3/3)

- Sum of absolute values (SOAV) optimization [5]

relax the ML method to convex optimization

by adding regularizer $\sum_{\ell=1}^{L} q_\ell \|\boldsymbol{s} - r_\ell \mathbf{1}\|_1$ ← parameter

$(\mathbf{1} = [1 \ 1 \ \cdots \ 1]^{\mathrm{T}})$

based on the fact that $\boldsymbol{x} - r_\ell \mathbf{1}$ has some zero elements (and sometimes becomes sparse) because $\boldsymbol{x} \in \{r_1, \dots, r_L\}^N$

SOAV optimization: $\underset{\boldsymbol{s} \in \mathbb{R}^N}{\text{minimize}} \ \frac{1}{2}\|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{s}\|_2^2 + \sum_{\ell=1}^{L} q_\ell \|\boldsymbol{s} - r_\ell \mathbf{1}\|_1$

data fidelity term — regularization term

SOAV optimization can take the probability distribution of unknown variables $p_\ell := \Pr(x_n = r_\ell) \ \ (\ell = 1, \dots, L)$ into consideration

[5] M. Nagahara, "Discrete signal reconstruction by sum of absolute values," IEEE Signal Process. Lett., vol. 22, no. 10, pp. 1575–1579, Oct. 2015.

Asymptotic SER of Box Relaxation Method

Asymptotic symbol error rate (SER) of box relaxation method has been studied via **convex Gaussian min-max theorem (CGMT)** [6], [7]

Ex. BPSK signals

estimate by box relaxation method

SER: $\frac{1}{N}\|\mathrm{sign}(\hat{\boldsymbol{x}}_{\mathrm{Box}}) - \boldsymbol{x}\|_0$

Assumption:
- measurement matrix $\boldsymbol{A}$: zero mean i.i.d. Gaussian
- noise vector $\boldsymbol{v}$: zero mean i.i.d. Gaussian

large system limit $M, N \to \infty$ $(M/N = \Delta)$

CDF of the standard Gaussian distribution

asymptotic SER: $1 - P\left(\frac{1}{\tau^*}\right)$

characterized by an optimization problem

[6] C. Thrampoulidis, E. Abbasi, and B. Hassibi, "Precise error analysis of regularized M-estimators in high dimensions," IEEE Trans. Inf. Theory, vol. 64, no. 8, pp. 5592–5628, Aug. 2018.
[7] C. Thrampoulidis, W. Xu, and B. Hassibi, "Symbol error rate performance of box-relaxation decoders in massive MIMO," IEEE Trans. Signal Process., vol. 66, no. 13, pp. 3377–3392, Jul. 2018.

Purpose of This Study

Only a few theoretical aspects are known for the SOAV optimization

$$\underset{s \in \mathbb{R}^N}{\text{minimize}} \ \frac{1}{2}\|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{s}\|_2^2 + \sum_{\ell=1}^{L} q_\ell \|\boldsymbol{s} - r_\ell \boldsymbol{1}\|_1$$

- ✦ How to tune the parameter q_ℓ ?
- ✦ How does the measurement ratio $\Delta = M/N$ affect the performance?

Purpose of This Study

analyze the **asymptotic performance** of discrete-valued vector reconstruction based on the SOAV optimization

① SOAV optimization —modify→ **Box-SOAV optimization**

② derive the asymptotic SER of Box-SOAV by using the CGMT framework

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Box-SOAV Optimization (1/2)

To make the analysis simpler,
we modify the SOAV optimization to **Box-SOAV optimization**

SOAV optimization

$$\underset{\boldsymbol{s}\in\mathbb{R}^N}{\text{minimize}} \quad \frac{1}{2}\|\boldsymbol{y}-\boldsymbol{A}\boldsymbol{s}\|_2^2 + \sum_{\ell=1}^{L} q_\ell \|\boldsymbol{s}-r_\ell \mathbf{1}\|_1$$

add box constraint $\boldsymbol{s} \in [r_1, r_L]^N$
(In usual, the performance does not change so much)

Box-SOAV optimization

$$\underset{\boldsymbol{s}\in[r_1,\, r_L]^N}{\text{minimize}} \quad \frac{1}{2}\|\boldsymbol{y}-\boldsymbol{A}\boldsymbol{s}\|_2^2 + \sum_{\ell=1}^{L} q_\ell \|\boldsymbol{s}-r_\ell \mathbf{1}\|_1$$

Box-SOAV Optimization (2/2)

Box-SOAV optimization can be solved by proximal splitting methods [8]

$$\underset{\boldsymbol{s}\in[r_1,\, r_L]^N}{\text{minimize}} \ \frac{1}{2}\|\boldsymbol{y}-\boldsymbol{A}\boldsymbol{s}\|_2^2 + \sum_{\ell=1}^{L} q_\ell \|\boldsymbol{s} - r_\ell \boldsymbol{1}\|_1$$

indicator function for $\boldsymbol{s} \in [r_1,\, r_L]^N$

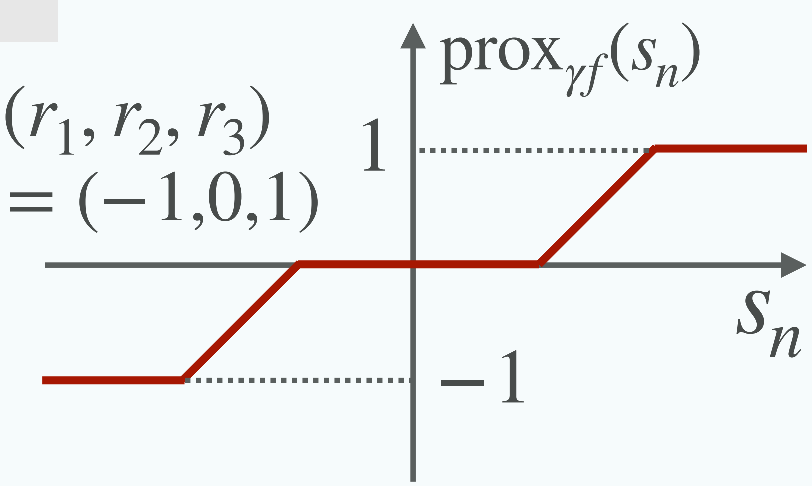
$$\underset{\boldsymbol{s}\in\mathbb{R}^N}{\text{minimize}} \ \frac{1}{2}\|\boldsymbol{y}-\boldsymbol{A}\boldsymbol{s}\|_2^2 + \underbrace{\sum_{\ell=1}^{L} q_\ell \|\boldsymbol{s} - r_\ell \boldsymbol{1}\|_1 + \mathcal{I}(\boldsymbol{s})}_{= f(\boldsymbol{s})}$$

$$\mathcal{I}(\boldsymbol{s}) = \begin{cases} 0 & (\boldsymbol{s} \in [r_1,\, r_L]^N) \\ \infty & (\boldsymbol{s} \notin [r_1,\, r_L]^N) \end{cases}$$

proximity operator $\text{prox}_{\gamma f}(\boldsymbol{s})$ $(\gamma > 0)$ can be easily calculated

$$\left(\text{prox}_{\gamma f}(\boldsymbol{s}) := \underset{\boldsymbol{u}\in\mathbb{R}^N}{\arg\min} \left\{ \gamma f(\boldsymbol{u}) + \frac{1}{2}\|\boldsymbol{u}-\boldsymbol{s}\|_2^2 \right\}\right)$$

Ex. $(r_1, r_2, r_3) = (-1, 0, 1)$



[8] P. L. Combettes and J.-C. Pesquet, "Proximal splitting methods in signal processing," in Fixed-point algorithms for inverse problems in science and engineering. Springer, 2011.

Main Result

SER: $\frac{1}{N}\|\mathcal{Q}(\hat{\boldsymbol{x}}) - \boldsymbol{x}\|_0$

$\mathcal{Q}$: quantization to the nearest r_ℓ

Assumption:

- ✦ $\boldsymbol{A}$: zero mean i.i.d. Gaussian
- ✦ $\boldsymbol{v}$: zero mean i.i.d. Gaussian

large system limit $M, N \to \infty$ $(M/N = \Delta)$

asymptotic SER:

$$1 - \sum_{\ell=1}^{L} p_\ell \left\{ P\left(\frac{\sqrt{\Delta}}{2\alpha^*}(r_{\ell+1} - r_\ell) + \frac{Q_{\ell+1}}{\beta^*} \right) - P\left(\frac{\sqrt{\Delta}}{2\alpha^*}(r_{\ell-1} - r_\ell) + \frac{Q_\ell}{\beta^*} \right) \right\}$$

P: CDF of the standard Gaussian distribution

- ✦ $Q_\ell = \left(\sum_{k=1}^{\ell-1} q_k\right) - \left(\sum_{k=\ell}^{L} q_k\right)$ $\quad (Q_1 = -\infty,\ Q_{L+1} = \infty,\ r_0 = -\infty,\ r_{L+1} = \infty)$
- ✦ $p_\ell = \Pr(x_n = r_\ell)$
- ✦ α^*, β^* : optimizer of problem $\max_{\beta>0} \min_{\alpha>0} F(\alpha, \beta)$

F: convex-concave function associated with Box-SOAV optimization

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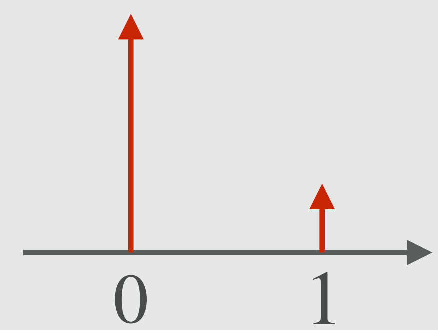
Example 1

Reconstruction of binary vector $\boldsymbol{x} \in \{0,1\}^N$

- measurement ratio: $\Delta = 0.75$
- distribution: $\Pr(x_n = 0) = 0.8$
  $\Pr(x_n = 1) = 0.2$
- SNR: 15 dB



Box-SOAV: $$\underset{\boldsymbol{s} \in [0,1]^N}{\text{minimize}} \ \frac{1}{2}\|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{s}\|_2^2 + q_1\|\boldsymbol{s}\|_1 + q_2\|\boldsymbol{s} - \mathbf{1}\|_1$$

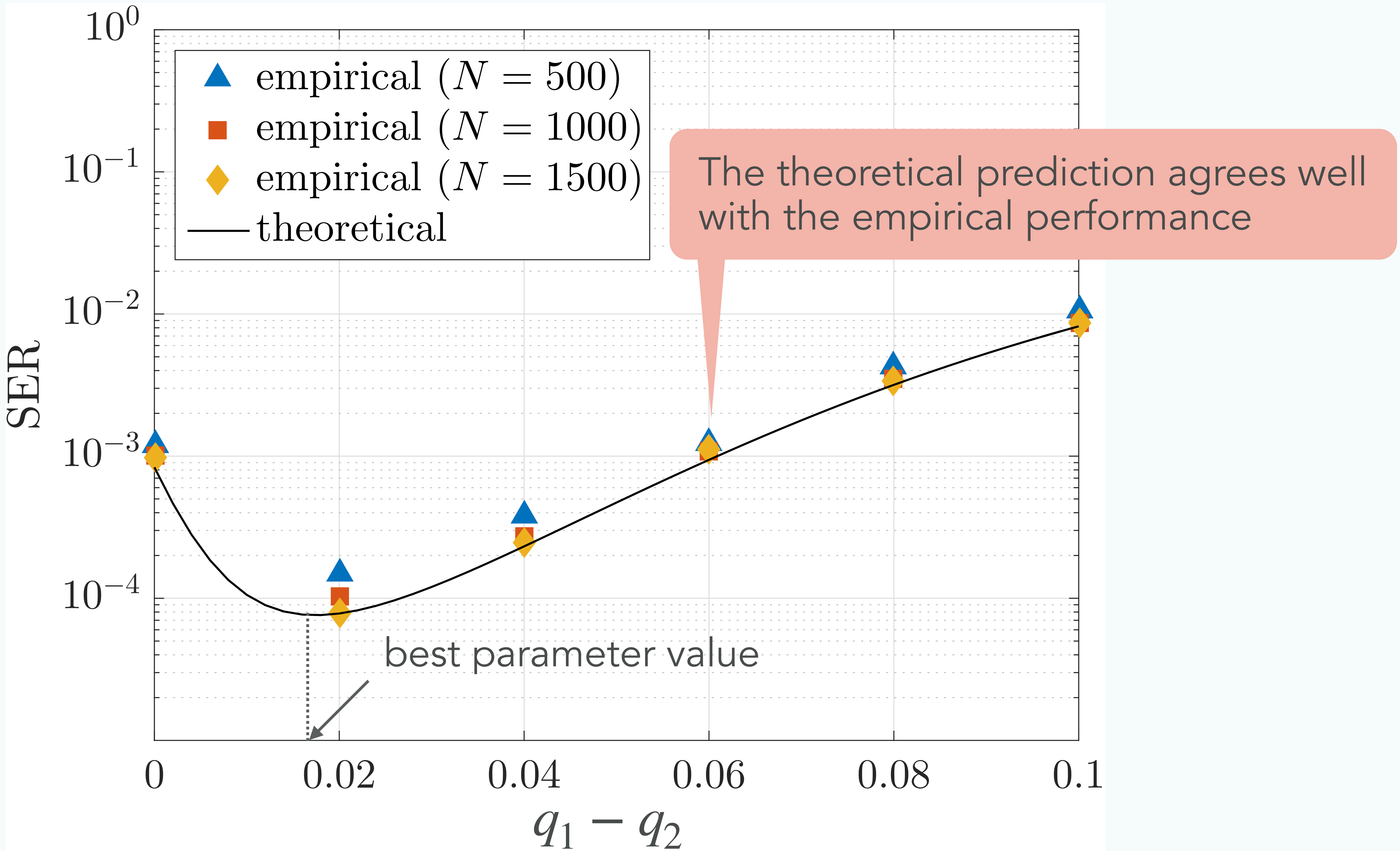
For $s \in [0,1]$,

$$q_1|s| + q_2|s-1| = q_1 s - q_2(s-1)$$
$$= (q_1 - q_2)s + (\text{const.})$$

$$\underset{\boldsymbol{s} \in [0,1]^N}{\text{minimize}} \ \frac{1}{2}\|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{s}\|_2^2 + (q_1 - q_2)\sum_{n=1}^{N} s_n$$

parameter

SER

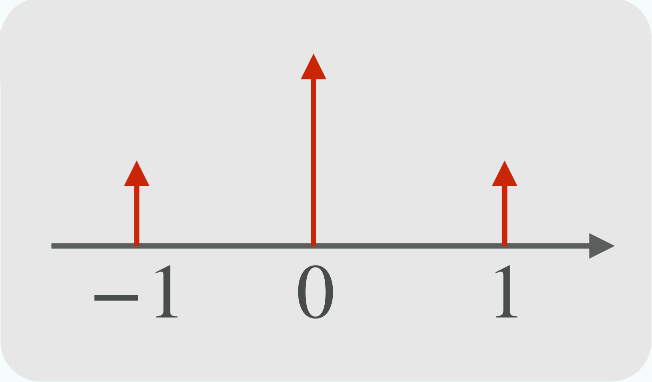

empirical ($N = 500$)
empirical ($N = 1000$)
empirical ($N = 1500$)
theoretical
The theoretical prediction agrees well with the empirical performance
best parameter value
SER
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
0
0.02
0.04
0.06
0.08
0.1
$q_1 - q_2$

Example 2

Reconstruction of discrete-valued vector $\boldsymbol{x} \in \{-1,0,1\}^N$

$(N = 1500)$

- distribution: $\Pr(x_n = -1) = 0.25$
  $\Pr(x_n = 0) = 0.5$
  $\Pr(x_n = 1) = 0.25$

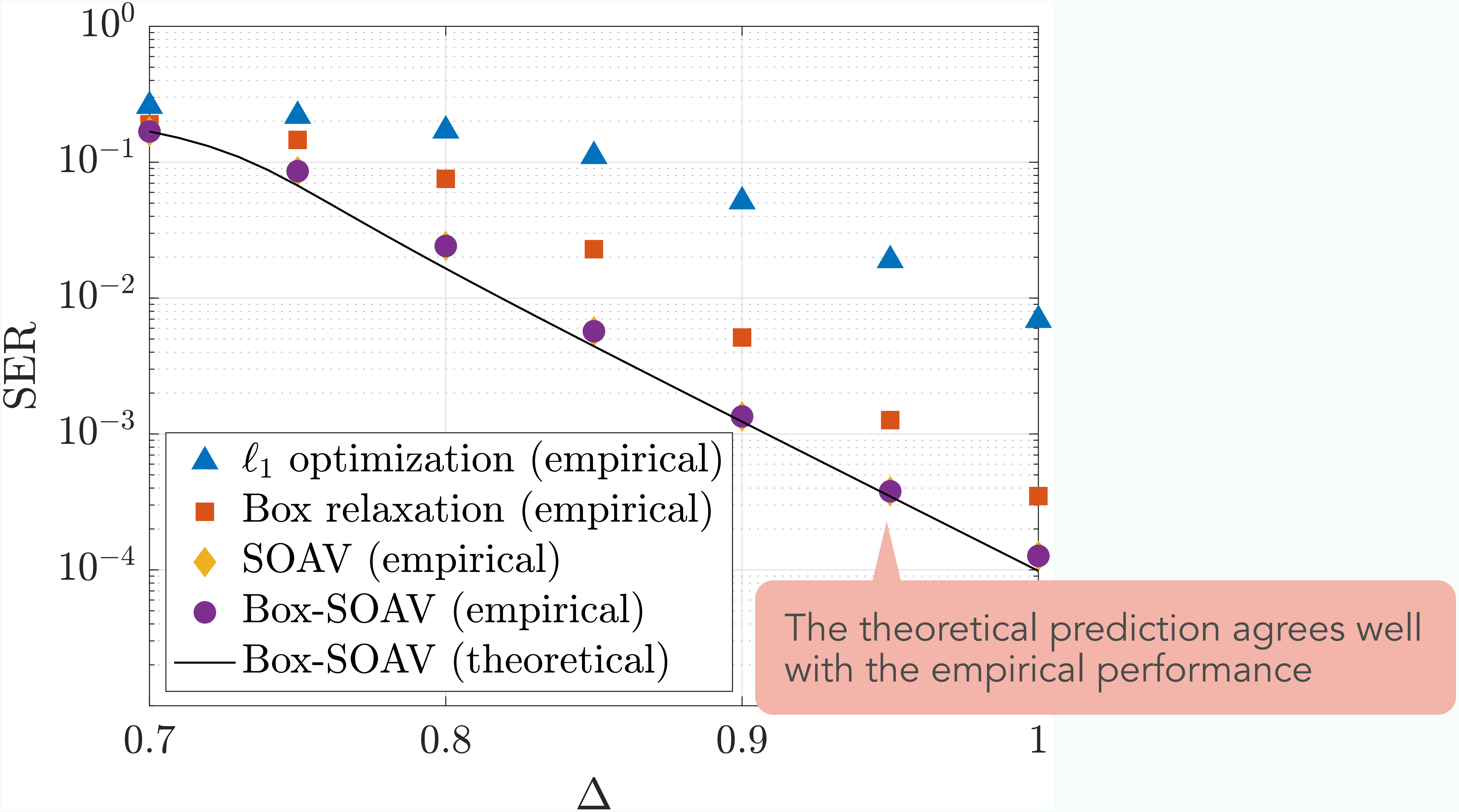


- SNR: 20 dB

ℓ_1 optimization	$\underset{\boldsymbol{s}\in\mathbb{R}^N}{\text{minimize}} \ \frac{1}{2}\|\boldsymbol{y}-\boldsymbol{A}\boldsymbol{s}\|_2^2 + \lambda\|\boldsymbol{s}\|_1$
Box relaxation	$\underset{\boldsymbol{s}\in[-1,1]^N}{\text{minimize}} \ \frac{1}{2}\|\boldsymbol{y}-\boldsymbol{A}\boldsymbol{s}\|_2^2$
SOAV	$\underset{\boldsymbol{s}\in\mathbb{R}^N}{\text{minimize}} \ \frac{1}{2}\|\boldsymbol{y}-\boldsymbol{A}\boldsymbol{s}\|_2^2 + q_1\|\boldsymbol{s}+\boldsymbol{1}\|_1 + q_2\|\boldsymbol{s}\|_1 + q_3\|\boldsymbol{s}-\boldsymbol{1}\|_1$
Box-SOAV	$\underset{\boldsymbol{s}\in[-1,1]^N}{\text{minimize}} \ \frac{1}{2}\|\boldsymbol{y}-\boldsymbol{A}\boldsymbol{s}\|_2^2 + q_1\|\boldsymbol{s}+\boldsymbol{1}\|_1 + q_2\|\boldsymbol{s}\|_1 + q_3\|\boldsymbol{s}-\boldsymbol{1}\|_1$

$$\lambda = 0.005,\ (q_1,\ q_2,\ q_3) = (1,\ 0.005,\ 1)$$

SER vs Measurement Ratio



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Conclusion

Summary of This Study

We have derived the theoretical asymptotic performance of the Box-SOAV optimization

1. SOAV optimization —add box constraint→ **Box-SOAV optimization**
2. derive the asymptotic SER of Box-SOAV by using the CGMT framework
3. compare the theoretical prediction and the empirical performance of the SOAV optimization and the Box-SOAV optimization

Future Work

- ✦ asymptotic distribution of estimates
- ✦ optimization of quantization

Appendix

α^*, β^*

- α^*, β^* : optimizer of problem $\max_{\beta>0} \min_{\alpha>0} F(\alpha, \beta)$ convex-concave function associated with Box-SOAV optimization

$$F(\alpha,\beta) = \frac{\alpha\beta\sqrt{\Delta}}{2} + \frac{\sigma_{\mathrm{v}}^2\beta\sqrt{\Delta}}{2\alpha} - \frac{1}{2}\beta^2 - \frac{\alpha\beta}{2\sqrt{\Delta}} + \frac{\beta\sqrt{\Delta}}{\alpha}\mathrm{E}\left[\mathrm{env}_{\frac{\alpha}{\beta\sqrt{\Delta}}f}\left(X + \frac{\alpha}{\sqrt{\Delta}}H\right)\right]$$

- σ_{v}^2 : noise variance
- $\mathrm{env}_{\frac{\alpha}{\beta\sqrt{\Delta}}f}(z) = \min_{u\in\mathbb{R}}\left\{\frac{\alpha}{\beta\sqrt{\Delta}}f(u) + \frac{1}{2}(u-z)^2\right\}$: Moreau envelope of $\frac{\alpha}{\beta\sqrt{\Delta}}f$
- X : random variable whose distribution is $\Pr(X = r_\ell) = p_\ell$
- H: standard Gaussian random variable