Performance Analysis of Discrete-Valued Vector Reconstruction Based on Box-Constrained Sum of L1 Regularizers

Ryo Hayakawa (Kyoto University, Japan)
Kazunori Hayashi (Osaka City University, Japan)
1. Introduction

2. Main Result

3. Simulation Results

4. Conclusion
Outline

1. **Introduction**
2. Main Result
3. Simulation Results
4. Conclusion
Discrete-Valued Vector Reconstruction

Reconstruction of a \textbf{discrete-valued} vector \(x \in \{r_1, \ldots, r_L\}^N (r_1 < \cdots < r_L)\) from its \textbf{underdetermined} linear measurement \(y = Ax + v \in \mathbb{R}^M (M < N)\)

\[y = A \begin{bmatrix} x \end{bmatrix} + v \]

\[\hat{x}\]

\textbf{Application}

\begin{itemize}
  \item overloaded MIMO signal detection [1]
  \item multiuser detection [2]
  \item faster-than-Nyquist signaling [3]
\end{itemize}


Reconstruction Methods (1/3)

**Linear minimum mean-square-error (LMMSE) approach**

- **Low complexity**
  - The performance is degraded in **underdetermined** problems

**Maximum likelihood (ML) approach**

perform exhaustive search to reconstruct \( \mathbf{x} \in \{r_1, \ldots, r_L\}^N \)

\[
\begin{align*}
\text{minimize} & \quad \frac{1}{2} \| \mathbf{y} - \mathbf{A}s \|^2_2 \\
\text{subject to} & \quad s \in \{r_1, \ldots, r_L\}^N
\end{align*}
\]

- **Good performance**
  - The computational complexity is prohibitive in **large-scale** problems

In **large-scale underdetermined** problems, we require a low-complexity method which can achieve reasonable performance
Reconstruction Methods (2/3)

Box relaxation method [4]

relax the ML method to convex optimization under the box constraint \( s \in [r_1, r_L]^N \) \((r_1 < \cdots < r_L)\)

\[
\begin{align*}
\text{minimize} & \quad \frac{1}{2} \| y - As \|^2_2 \\
\text{subject to} & \quad s \in [r_1, r_L]^N
\end{align*}
\]

(binary phase shift keying)

Ex. For the reconstruction of BPSK signals \( x \in \{-1, 1\}^N \), box relaxation problem is given by

\[
\begin{align*}
\text{minimize} & \quad \frac{1}{2} \| y - As \|^2_2 \\
\text{subject to} & \quad s \in [-1, 1]^N
\end{align*}
\]

**Sum of absolute values (SOAV) optimization [5]**

relax the ML method to convex optimization

by adding regularizer \( \sum_{\ell=1}^{L} q_{\ell} \| s - r_{\ell}1 \|_1 \)

\((1 = [1 \ 1 \ \ldots \ 1]^T)\)

based on the fact that \( x - r_{\ell}1 \) has some zero elements

(and sometimes becomes sparse) because \( x \in \{r_1, \ldots, r_L\}^N \)

SOAV optimization: minimize

\[
\frac{1}{2} \| y - As \|_2^2 + \sum_{\ell=1}^{L} q_{\ell} \| s - r_{\ell}1 \|_1
\]

data fidelity term  \hspace{5cm} \text{regularization term}

SOAV optimization can take the probability distribution of unknown variables

\( p_{\ell} := \Pr(x_n = r_{\ell}) \)  \((\ell = 1, \ldots, L)\) into consideration

---

Asymptotic symbol error rate (SER) of box relaxation method has been studied via convex Gaussian min-max theorem (CGMT) [6], [7].

Ex. BPSK signals estimate by box relaxation method

\[
\text{SER: } \frac{1}{N} \| \text{sign}(\hat{x}_\text{Box}) - x \|_0
\]

Assumption:
- measurement matrix \( A \): zero mean i.i.d. Gaussian
- noise vector \( \nu \): zero mean i.i.d. Gaussian

large system limit \( M, N \to \infty \) \( (M/N = \Delta) \)

CDF of the standard Gaussian distribution characterized by an optimization problem

asymptotic SER: 

\[
1 - P \left( \frac{1}{\tau^*} \right)
\]


Only a few theoretical aspects are known for the SOAV optimization

\[
\min_{s \in \mathbb{R}^N} \frac{1}{2} \|y - As\|_2^2 + \sum_{\ell=1}^L q_\ell \|s - r_\ell 1\|_1
\]

- How to tune the parameter \( q_\ell \)?
- How does the measurement ratio \( \Delta = M/N \) affect the performance?

**Purpose of This Study**

analyze the **asymptotic performance** of discrete-valued vector reconstruction based on the SOAV optimization

1. SOAV optimization **modify** Box-SOAV optimization
2. derive the asymptotic SER of Box-SOAV by using the CGMT framework
Outline

1. Introduction
2. Main Result
3. Simulation Results
4. Conclusion
To make the analysis simpler, we modify the SOAV optimization to **Box-SOAV optimization**

**SOAV optimization**

\[
\minimize_{s \in \mathbb{R}^N} \frac{1}{2} \| y - As \|_2^2 + \sum_{\ell=1}^L q_{\ell} \| s - r_{\ell} 1 \|_1
\]

add box constraint \( s \in [r_1, r_L]^N \)
(In usual, the performance does not change so much)

**Box-SOAV optimization**

\[
\minimize_{s \in [r_1, r_L]^N} \frac{1}{2} \| y - As \|_2^2 + \sum_{\ell=1}^L q_{\ell} \| s - r_{\ell} 1 \|_1
\]
Main Result

Box-SOAV Optimization (2/2)

Box-SOAV optimization can be solved by proximal splitting methods [8]

\[
\begin{align*}
\text{minimize} & \quad \frac{1}{2} \|y - As\|_2^2 + \sum_{\ell=1}^{L} q_\ell \|s - r_\ell 1\|_1 \\
\text{subject to} & \quad s \in [r_1, r_L]^N
\end{align*}
\]

\[
\begin{align*}
\text{minimize} & \quad \frac{1}{2} \|y - As\|_2^2 + \sum_{\ell=1}^{L} q_\ell \|s - r_\ell 1\|_1 + \mathcal{J}(s) \\
= f(s)
\end{align*}
\]

proximity operator \( \text{prox}_{\gamma f}(s) \) (\( \gamma > 0 \))

Ex. \((r_1, r_2, r_3) = (-1, 0, 1)\)

Main Result

SER: \( \frac{1}{N} \| Q(\hat{x}) - x \|_0 \)

quantization to the nearest \( r_\ell \)

large system limit \( M, N \to \infty \)

\( (M/N = \Delta) \)

asymptotic SER:

\[
1 - \sum_{\ell=1}^{L} p_\ell \left\{ P \left( \frac{\sqrt{\Delta}}{2\alpha^*} (r_{\ell+1} - r_\ell) + \frac{Q_{\ell+1}}{\beta^*} \right) - P \left( \frac{\sqrt{\Delta}}{2\alpha^*} (r_{\ell-1} - r_\ell) + \frac{Q_\ell}{\beta^*} \right) \right\}
\]

\( Q_\ell = \left( \sum_{k=1}^{\ell-1} q_k \right) - \left( \sum_{k=\ell}^{L} q_k \right) \) \( (Q_1 = -\infty, Q_{L+1} = \infty, r_0 = -\infty, r_{L+1} = \infty) \)

\( p_\ell = \text{Pr}(x_n = r_\ell) \)

\( \alpha^*, \beta^* : \text{optimizer of problem} \ \max_{\beta > 0} \min_{\alpha > 0} F(\alpha, \beta) \)

CDF of the standard Gaussian distribution

Assumption:

\( \mathbf{A} : \text{zero mean i.i.d. Gaussian} \)

\( \mathbf{v} : \text{zero mean i.i.d. Gaussian} \)
Outline

1. Introduction
2. Main Result
3. Simulation Results
4. Conclusion
Reconstruction of binary vector $x \in \{0,1\}^N$

- **measurement ratio**: $\Delta = 0.75$
- **distribution**: $\Pr(x_n = 0) = 0.8$
  $\Pr(x_n = 1) = 0.2$
- **SNR**: 15 dB

**Box-SOAV:**

\[
\min_{s \in \{0,1\}^N} \quad \frac{1}{2} \|y - As\|_2^2 + q_1 \|s\|_1 + q_2 \|s - 1\|_1
\]

For $s \in [0,1]$, 
$q_1 |s| + q_2 |s - 1| = q_1 s - q_2 (s - 1) = (q_1 - q_2) s + (\text{const.})$
The theoretical prediction agrees well with the empirical performance.
Example 2

Reconstruction of discrete-valued vector $x \in \{-1,0,1\}^N$ ($N = 1500$)

- distribution: $\Pr(x_n = -1) = 0.25$
  $\Pr(x_n = 0) = 0.5$
  $\Pr(x_n = 1) = 0.25$

- SNR: 20 dB

<table>
<thead>
<tr>
<th>Optimization Type</th>
<th>Objective Function</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\ell_1$ optimization</td>
<td>$\min_{s \in \mathbb{R}^N} \frac{1}{2} |y - As|_2^2 + \lambda |s|_1$</td>
</tr>
<tr>
<td>Box relaxation</td>
<td>$\min_{s \in [-1,1]^N} \frac{1}{2} |y - As|_2^2$</td>
</tr>
<tr>
<td>SOAV</td>
<td>$\min_{s \in \mathbb{R}^N} \frac{1}{2} |y - As|_2^2 + q_1 |s + 1|_1 + q_2 |s|_1 + q_3 |s - 1|_1$</td>
</tr>
<tr>
<td>Box-SOAV</td>
<td>$\min_{s \in [-1,1]^N} \frac{1}{2} |y - As|_2^2 + q_1 |s + 1|_1 + q_2 |s|_1 + q_3 |s - 1|_1$</td>
</tr>
</tbody>
</table>

$\lambda = 0.005$, $(q_1, q_2, q_3) = (1, 0.005, 1)$
The theoretical prediction agrees well with the empirical performance.
Outline

1. Introduction
2. Main Result
3. Simulation Results
4. Conclusion
Conclusion

Summary of This Study

We have derived the theoretical asymptotic performance of the Box-SOAV optimization.

1. SOAV optimization → add box constraint → Box-SOAV optimization

2. Derive the asymptotic SER of Box-SOAV by using the CGMT framework.

3. Compare the theoretical prediction and the empirical performance of the SOAV optimization and the Box-SOAV optimization.

Future Work

✦ asymptotic distribution of estimates
✦ optimization of quantization
Appendix

\( \alpha^*, \beta^* \)

- \( \alpha^*, \beta^* \): optimizer of problem \( \max_{\beta > 0} \min_{\alpha > 0} F(\alpha, \beta) \) convex-concave function associated with Box-SOAV optimization

\[
F(\alpha, \beta) = \frac{\alpha \beta \sqrt{\Delta}}{2} + \frac{\sigma_v^2 \beta \sqrt{\Delta}}{2\alpha} - \frac{1}{2} \beta^2 - \frac{\alpha \beta}{2\sqrt{\Delta}} + \frac{\beta \sqrt{\Delta}}{\alpha} E \left[ \text{env}_{\frac{\alpha}{\beta \sqrt{\Delta}}} f \left( X + \frac{\alpha}{\sqrt{\Delta}} H \right) \right]
\]

- \( \sigma_v^2 \): noise variance

- \( \text{env}_{\frac{\alpha}{\beta \sqrt{\Delta}}} f(z) = \min_{u \in \mathbb{R}} \left\{ \frac{\alpha}{\beta \sqrt{\Delta}} f(u) + \frac{1}{2} (u - z)^2 \right\} \): Moreau envelope of \( \frac{\alpha}{\beta \sqrt{\Delta}} f \)

- \( X \): random variable whose distribution is \( \Pr(X = r_\ell) = p_\ell \)

- \( H \): standard Gaussian random variable