Disjunct Matrices for Compressed Sensing

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Outline

1. Motivation
2. Contributions
3. Basics of Disjunct matrices
4. Relation with Spark and Mutual Coherence
5. Recovery of Sparse Signals Using a Binary Matrix
   - Recovery of all sparse signals using Binary Matrix
   - Recovery of Almost All Sparse Signals Using a Binary Matrix
6. Disjunctness of a Sparse Matrix
7. Recovery of Sparse Signals Using a Sparse Matrix
   - Recovery of All Sparse Signals Using a Sparse Matrix
   - Recovery of almost all sparse signals using Sparse matrix
8. Simulation Results
   - Run-time Comparison
9. Future Work
Basic pursuit (BP) & orthogonal matching pursuit (OMP): polynomial complexity in problem dimension
- Impractical and expensive in high dimensional settings
- Verifying conditions based on spark and RIP is not easy
  - Hence, in practice, it remains unknown whether a given instantiation of the measurement matrix satisfies these properties

Goal
Identify a property of a matrix that is easy to verify and also supports low computational complexity sparse recovery algorithms, while perhaps requiring a larger number of measurements for success.
Contributions

- We connect non-adaptive group testing and compressed sensing
  - Disjunctness property of binary matrices is also very useful in recovering sparse signals
- We exploit the disjunctness property to present an ultra-low complexity algorithm for identifying the support as well as recover the nonzero coefficients of the sparse signal
  - Non-iterative algorithm, very fast
- We extend the disjunctness property of a binary matrix to sparse matrices. We show that a similar non-iterative and fast sparse recovery algorithm is possible
The set \( \{1, 2, \ldots, n\} \) is denoted by \([n]\).

The \( i \)-th entry of \( x \) is denoted by \( x_i \).

\( \Phi(:, i) \) and \( \Phi(j,:) \) denote the \( i \)-th column and \( j \)-th row of \( \Phi \), respectively, and \( \Phi(j, i) \) denotes the \((j, i)\)th entry of \( \Phi \).

The support of \( x \) is \( \{i : x_i \neq 0\} \), denoted by \( \text{supp}(x) \).

Let \( S \subset [n] \), then \( x_S \triangleq (x_i)_{i \in S} \) and \( \Phi_S \triangleq (\Phi(:, i))_{i \in S} \).
Disjunct Matrix

**Definition 1**

An $m \times M$ binary matrix $\Phi$ is called $t$-disjunct if the support of any column is not contained in the union of the supports of any other $t$ columns.

**Implications:**

- If we take a submatrix $\Phi_S$ with $|S| = t + 1$, then for $i \in [t + 1]$, there exists $j_i$ such that $\Phi_S(j_i, i) = 1$ and $\Phi_S(j_i, l) = 0$ for all $l \in [t + 1] \setminus i$
- This observation will be crucial for non-iterative recovery of almost all sparse signals

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**Definition 2**

A matrix $\Phi$ is $t^e$-disjunct if, given any $t + 1$ columns of $\Phi$ with one designated column, there are $e + 1$ rows with a 1 in the designated column and a 0 in each of the other $t$ columns.

**Implications:**

- If we take a submatrix $\Phi_S$ with $|S| = t + 1$, then for $i \in [t + 1]$, there exists $j_{i1}^1, \ldots, j_{ie+1}^i$ such that $\Phi_S(j_{id}^i, i) = 1$ and $\Phi_S(j_{id}^i, l) = 0$ for all $l \in [t + 1] \setminus i$ and $d = 1, \ldots, e + 1$.
- We exploit this property for recovering all signals with a given max. sparsity level.
Theorem 1

Let $\Phi$ be a $m \times M$ matrix with each column containing $q$ ones and the overlap (i.e., the size of the intersection of the supports) between any two distinct columns is at most $r$. Then $\Phi$ is $\left\lfloor \frac{q-1}{r} \right\rfloor$-disjunct.
Relation with Spark

Definition 3

The spark of a matrix is the smallest number of linearly dependent columns in the matrix.

- Necessary and sufficient condition for uniqueness
  - If $\text{spark}(\Phi) = k$, sparse vectors with up to $k/2$ nonzero entries (and no more) can be uniquely recovered from $y = \Phi x$.

Theorem 2

The spark of a $t$-disjunct matrix is at least $t + 1$.

Proof 1

Follows from the definition of a disjunct matrix.
Definition 4

The mutual coherence $\mu_\Phi$ of $\Phi$ is the maximum absolute inner product between any two distinct normalized columns of $\Phi$.

Theorem 3

A matrix $\Phi$ containing the same number of ones in each column is $(\lfloor \mu_\Phi^{-1} \rfloor - 1)$-disjunct.

Proof 2

Follows from the fact that if each column of $\Phi$ contains $q$ ones and the overlap between any two columns is at most $r$, then its mutual coherence $\mu_\Phi \leq \frac{r}{q}$.
Suppose $\Phi(:, i)$ contains $q_i$ ones for $i \in [M]$, $q_{\min} \triangleq \min\{q_1, \ldots, q_M\}$, and that the overlap between any two distinct columns is at most $r_{\max}$.

**Theorem 4**

$\Phi$ is $t^e$-disjunct for any $t < \left\lfloor \frac{q_{\min}}{r_{\max}} \right\rfloor$ and $e + 1 \geq q_{\min} - tr_{\max}$

**Theorem 5**

*Let $\Phi$ be a binary matrix with every column containing at least $q_{\min}$ ones and with the overlap between any two distinct columns at most $r_{\max}$. Then any $\left\lfloor \frac{q_{\min}}{2r_{\max}} \right\rfloor$-sparse vector can be uniquely recovered from $y = \Phi x$*
Proof (and a fast recovery algorithm)

Support recovery

$$S = \{j : |\text{supp}(\Phi(:, i)) \cap \text{supp}(y)| > \frac{q_{\text{min}}}{2} \}$$ is the support of $x$.

Non-zero coefficient recovery

Step 1: As $\Phi$ is $t^e$-disjunct for some $t < \left\lfloor \frac{q_{\text{min}}}{r_{\text{max}}} \right\rfloor$ and $e \geq q_{\text{min}} - tr_{\text{max}} - 1$, it is also $\left\lfloor \frac{q_{\text{min}}}{2r_{\text{max}}} \frac{q_{\text{min}}}{2} \right\rfloor$-disjunct.

Step 2: As a result, whenever $s \in S$, for $\Phi_S(:, s)$ there exist $j_s^1, \ldots, j_s^{e+1}$ rows such that $\Phi_S(j_s^d, s) = 1$ and $\Phi_S(j_s^d, l) = 0$ for $l \in S \setminus s$ and $d = 1, \ldots, e + 1$.

Step 3: Thus, we can directly recover

$$x_s = \begin{cases} y_{j_s^d}, & d = 1, \ldots, e + 1 \quad \text{if } i \in S \\ 0, & \text{otherwise.} \end{cases}$$ (1)
Recovery of Almost All Sparse Signals Using a $t$-Disjunct Binary Matrix

- **Assumption**: $y_j = \sum_{l \in \text{supp}(\Phi_S(j,:) )} x_l \neq 0, \ \forall \ j \in [m]$
- This holds (a) with probability 1 if $x$ is drawn from a generic random model; and (or?) (b) $x$ is a non negative sparse signal

**Support recovery**

$$S = [M] \setminus \bigcup_{j: y_j = 0} \text{supp}(\Phi(j,:))$$

**Non-zero coefficient recovery**

**Step-1**: As $\Phi$ is $t$-disjunct, for $i \in [k]$, there exists $j_i$ such that $\Phi_S(j_i, i) = 1$ and $\Phi_S(j_i, l) = 0$ for all $l \in [k] \setminus i$

**Step 2**: Set

$$x_i = \begin{cases} y_{ji}, & \text{if } i \in S \\ 0, & \text{otherwise} \end{cases}$$
Definition 5

An $m \times M$ sparse matrix $\Phi$ is said to be $t$-disjunct if the support of any column is not contained in the union of the supports of any $t$ other columns.

- Let $\Phi$ be a sparse matrix where $\Phi(:, i)$ contains $q_i$ non-zeros for $i \in [M]$ with $q_{\text{min}} \triangleq \min\{q_1, \ldots, q_M\}$
- Let the cardinality of the intersection between support of any two distinct columns be at most $r_{\text{max}}$

Theorem 6

$\Phi$ is $t^e$-disjunct if $t < \left\lfloor \frac{q_{\text{min}}}{r_{\text{max}}} \right\rfloor$ and $e + 1 \geq q_{\text{min}} - tr_{\text{max}}$. 
Recovery of All Sparse Signals Using a Sparse Matrix

Consider the linear system \( y = \Phi x \), where \( k < \frac{q_{\min}}{2r_{\max}} \)

Support recovery
\[
S = \{ j : |supp(\Phi(:, i)) \cap supp(y)| > \frac{q_{\min}}{2} \}.
\]

Non-zero coefficient recovery

(Step 1) and (Step 2): same as the binary case

Step 3: Set
\[
x_s = \begin{cases} 
\frac{y_{jd}}{\Phi_s(j_d, s)}, & d = 1, \ldots, e + 1 \quad \text{if } i \in S \\
0, & \text{otherwise}
\end{cases}
\]
Recovery of almost all sparse signals using Sparse matrix

- **Assumption**: \( y_j = \sum_{l \in \text{supp}(\Phi_S(j,:))} \Phi(j, l)x_l \neq 0, \forall j \in [m] \).
- This holds for same conditions as given for binary matrices.
- Consider the linear system \( y = \Phi x \), where \( \Phi \) is \( t \)-disjunct and \( k < t + 1 \).

**Support recovery**

\[ S = \{ i : \text{supp}(\Phi(:, i)) \subseteq \text{supp}(y) \} \]

**Non-zero coefficient recovery**

**Step 1**: same as in binary case.

**Step 2**: Now set

\[ x_i = \begin{cases} 
\frac{y_{ji}}{\Phi(j_i, i)}, & \text{if } i \in S \\
0, & \text{otherwise.}
\end{cases} \quad (4) \]
We use the binary sensing matrix $\Phi$ of size $q^2 \times q^{r+1}$ constructed by Devore\textsuperscript{4} for $q$ being prime power and $r > 1$.

Every column of $\Phi$ has $q$ ones and the overlap between any two distinct columns is at most $r$.

$\Phi$ is $\lfloor \frac{q-1}{r} \rfloor$-disjunct and $t^e$-disjunct with $t < \lfloor \frac{q}{r} \rfloor$ and $e + 1 \geq q - tr$

As an example, we take $\Phi$ of size $(29)^2 \times (29)^3$ Therefore, $\Phi$ is 14–disjunct and also $7^{14}$-disjunct (i.e., $t = 7$, $e = 14$) and $\mu_\Phi \leq \frac{2}{29}$

We consider sparsity $k \leq 33$. For each $k$, we generate 1000 random $k$–sparse vectors. Our algorithm recovers sparse vectors with $k = 15$ in all 1000 trials, as expected. Further, the algorithm can recover $x$ with much higher sparsity, up to $k = 33$, in all 1000 trials. An existing non-iterative sparse recovery algorithm\(^5\) can recover the unknown sparse vector $x$ only up to sparsity 7 exactly in all 1000 trials. Beyond $k = 9$, it fails to recover even a single unknown sparse vector. This is because the existing algorithm requires $4k < q$, i.e., $k < 8$, in order to ensure that each nonzero entry in $x$ occurs at least $q/2$ times in $y$.

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Run-time Comparison

Figure: Average runtime comparison between Our proposed method, OMP and the existing non-iterative algorithm for matrix size $(29)^2 \times (29)^3$
Future Work

- Deriving bounds on the number of rows required for the measurement matrix to satisfy $t$-disjunctness
- Sparse signal recovery guarantees for disjunct matrices in noisy measurement settings.
THANK YOU!