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Reconstruction of FRI Signals using Deep Neural Network Approaches

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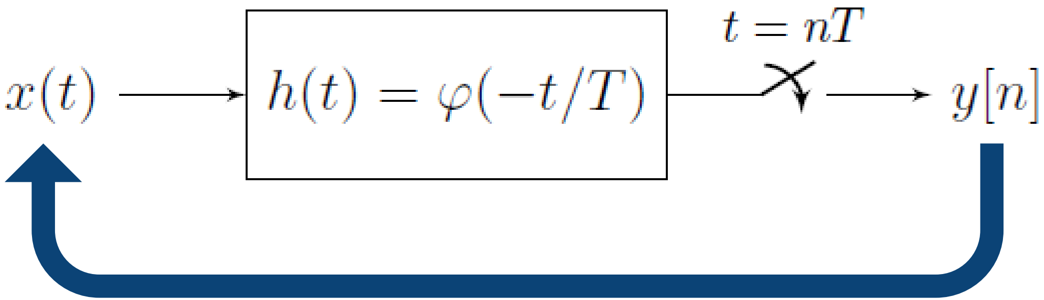
Problem Statement

Continuous Time Signal

Sampling Filter

Discrete Time Signal



Reconstruction

Finite Rate of Innovation (FRI)

- Classical sampling theory
  - Perfect reconstruction is possible when $x(t)$ is bandlimited
  - Sampling frequency $1/T$ is twice the bandwidth of the input signal
  - Sampling kernel $\varphi(t)$ is a sinc function

- FRI sampling theory
  - Extended to classes of non-bandlimited signals with finite number of degrees of freedom per unit time [1]
  - Perfect reconstruction is possible given appropriate sampling kernel choices

[1] M. Vetterli, P. Marziliano, and T. Blu, “Sampling signals with finite rate of innovation,” IEEE Transactions on Signal Processing, vol. 50, no. 6, pp. 1417–1428, Jun. 2002.

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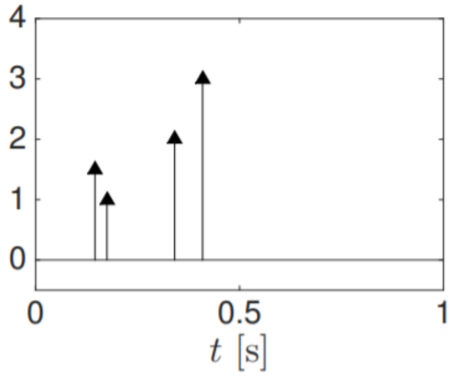
Example of FRI Signal

- Stream of K Diracs:

$$x(t) = \sum_{k=0}^{K-1} a_k \delta(t - t_k)$$



  - $2K$ rate of innovation
  - Many phenomena can be modelled as the convolution of a pulse shape with a stream of Diracs

Choices of Sampling Kernel φ(t)

- Satisfy generalised Strang-Fix conditions [2]
  - Able to reproduce exponential polynomials

$$\sum_{n\in\mathbb{Z}} c_{m,n,r}\varphi(t-n) = t^r e^{j\omega_m t} \text{ with } \omega_m = \omega_0 + m\lambda$$

  - Polynomial reproducing function ($m = 0$, e.g. B-spline)
  - Exponential reproducing function ($r = 0$, e.g. E-spline)

[2] G. Strang and G. Fix, “A Fourier analysis of the finite element variational method,” in Constructive Aspects of Functional Analysis. Springer Berlin Heidelberg, 2011, pp. 793–840.

Classical FRI Methods

- Samples of a stream of Diracs $\{y[n]\}_{n=0}^{N-1}$ can be written as

$$y[n] = \left\langle x(t), \varphi\left(\frac{t}{T} - n\right)\right\rangle = \sum_{k=0}^{K-1} a_k \varphi\left(\frac{t_k}{T} - n\right) \text{ for } n = 0, 1, \ldots, N-1$$

- Mapping the samples to sum of exponentials

$$s[m] = \sum_{n=0}^{N-1} c_{m,n} y[n] = \sum_{k=0}^{K-1} a_k \sum_{n\in\mathbb{Z}} c_{m,n} \varphi\left(\frac{t_k}{T} - n\right)$$

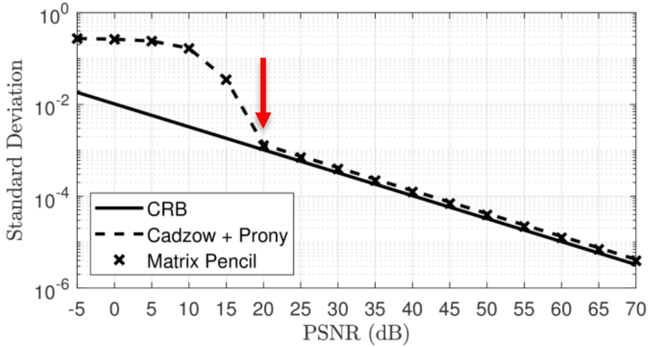
$$= \sum_{k=0}^{K-1} \underbrace{a_k e^{j\omega_0 t_k/T}}_{b_k} \Big(\underbrace{e^{j\lambda t_k/T}}_{u_k}\Big)^m = \sum_{k=0}^{K-1} b_k u_k^m \quad \text{for } m = 0, 1, \ldots, P$$

→ Spectral Estimation **(Non-linear w.r.t. locations)**

Classical FRI Methods

- Spectral estimation can be solved by SVD-based subspace methods (e.g. Prony's method [3] and Matrix Pencil method [4])

- When $\{y[n]\}_{n=0}^{N-1}$ is corrupted by additive white Gaussian noise
  - The performance follows Cramér-Rao bound (CRB) at high PSNR
  - Breaks down when PSNR drops below a certain level

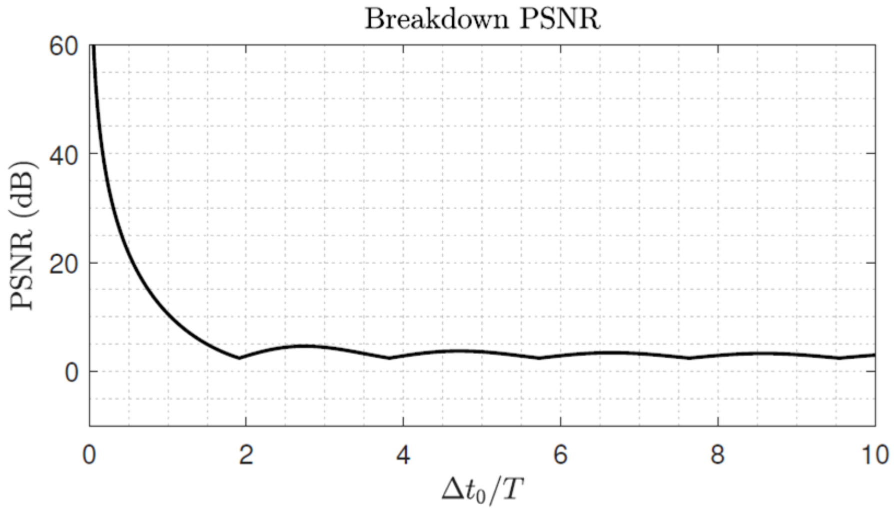


→ Develop methods that give more reliable estimations at low PSNR region while achieving near optimal performances at high PSNR region

[3] R. Prony, "Essai experimental et analytique," J. de l'Ecole Polytechnique, vol. 1, pp.24-76, 1795.
[4] Y. Hua and T. K. Sarkar, "Matrix Pencil Method for Estimating Parameters of Exponentially Damped/Undamped Sinusoids in Noise," IEEE Transactions on Acoustics, Speech, and Signal Processing, vol. 38, no. 5, pp. 814-824, May 1990.

Breakdown PSNR

- Conjectured to be the necessary condition for confusion between noise and signal subspaces to occur (Subspace swap event) [4]
- For a stream of 2 Diracs of equal amplitude,



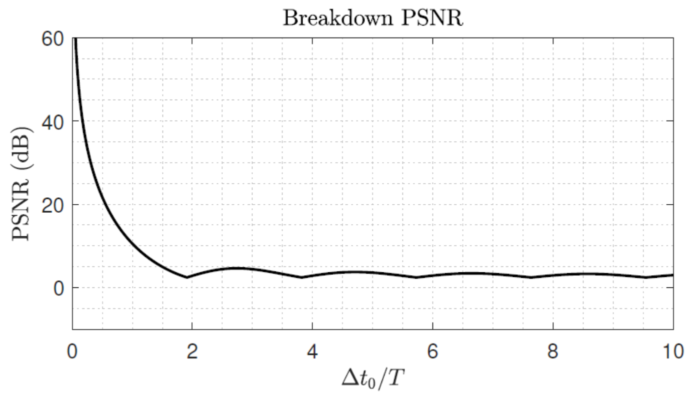
$$\mathrm{PSNR} < 10\log_{10}\frac{8\left(\frac{P}{2}+1\right)\ln\left(\frac{P}{2}+1\right)}{\left(\frac{P}{2}+1-\frac{\sin(\frac{\lambda}{2}(\frac{P}{2}+1)\Delta t_0/T)}{\sin(\frac{\lambda}{2}\Delta t_0/T)}\right)^2}$$

The smaller the distance between the two neighboring Diracs ($\frac{\Delta t_k}{T}$ with $\Delta t_k = t_{k+1} - t_k$), the higher the breakdown SNR will be

[5] X. Wei and P. L. Dragotti, “Guaranteed performance in the FRI setting,” IEEE Signal Processing Letters, vol. 22, no. 10, pp. 1661-1665, 2015.

DNN-based Approaches

- Explore an alternative approach to solve FRI problem to alleviate the subspace swap problem



- DNN-based methods have achieved state-of-the-art performances on many signal processing problem by learning from large amount of training data pairs

  → Exploit the advantage of DNN and existing training data

Direct Inference: Motivation

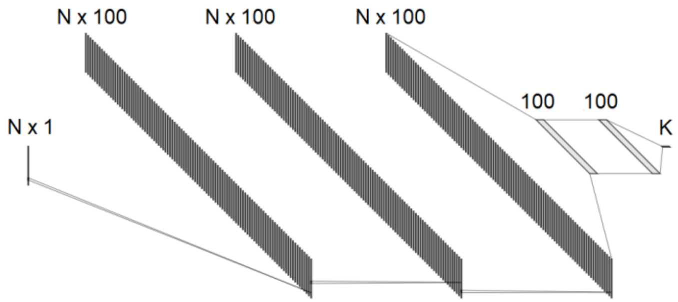
- Inferring locations $\{\hat{t}_k\}_{k=0}^{K-1}$ from noisy samples $\{\tilde{y}[n]\}_{n=0}^{N-1}$ directly using DNN

- Bypass the classical subspace methods
  - May reduce the occurrence of inherent subspace swap event

- Does not require any explicit information about the sampling kernel $\varphi(t)$
  - Implicitly learn the relationship from training the network with large amount of data from the same sampling kernel

Direct Inference: Implementation

- Network Structure:
  - 3 Convolutional Layers, followed by 3 Fully Connected Layers of size $100, 100, K$
  - Rectified Linear Unit (ReLU) as activation between each layer
  - Mean-squared error $\sum_{k=0}^{K-1}(\hat{t}_k - t_k)^2$



Denoising Samples: Motivation

- Firstly denoise noisy samples $\{\tilde{y}[n]\}_{n=0}^{N-1}$ using DNN, then apply classical FRI methods to retrieve $\{\hat{t}_k\}_{k=0}^{K-1}$
- Lower the breakdown PSNR without significantly altering the performance in the low noise regime
  - Subspace swap event may remain as it is inherent to subspace-based reconstruction methods

Denoising Samples: Implementation

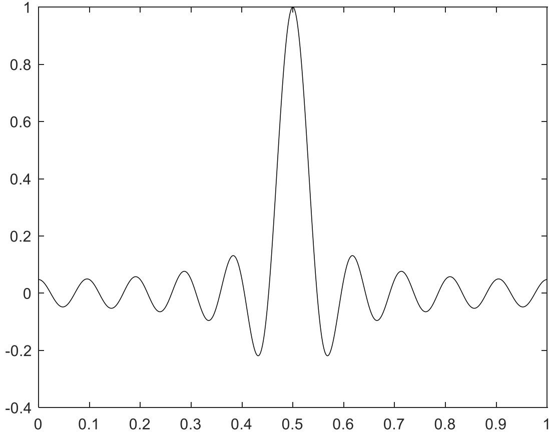
- Network Structure:
  - Similar to the direct inference approach
  - 3 Convolutional Layers with 100 filters, followed by 3 Fully Connected Layers with size $100N, 20N, N$
  - Rectified Linear Unit (ReLU) as activation
  - Mean-squared error $\sum_{n=0}^{N-1}(\hat{y}[n] - y[n])^2$

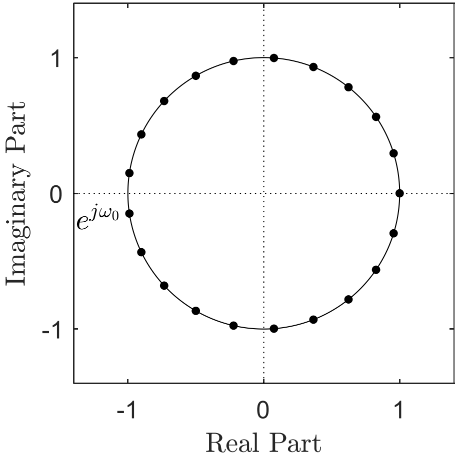
Simulation Setup

- Task: Reconstructing a stream of 2 Diracs with $t_k \in [-0.5,0.5)$ and $a_k \in \mathbb{R}^+$
- Number of samples $N = 21$, Sampling period $T = \frac{1}{N} = \frac{1}{21}$
- DNN trained for each PSNR $\in [-5, 70]$ dB with a step of 5 dB
- $100{,}000$ training data with $t_k \sim \mathcal{U}(-0.5,0.5)$ and $a_k \sim \mathcal{U}(0.5,10)$, where $\mathcal{U}(a, b)$ denotes uniform distribution between a and b.

Simulation Setup

- Optimal sampling kernel for subspace methods
  - An exponential reproducing kernel of maximum order and minimum-support (e-MOMS) that can reproduce $P+1=N$ exponentials evenly spaced around the unit circle





Evaluation Method

- Metric: Standard Deviation $$\sqrt{\frac{\sum_{i=0}^{I-1}\left(\hat{t}_k^{(i)}-t_k\right)^2}{I}}$$

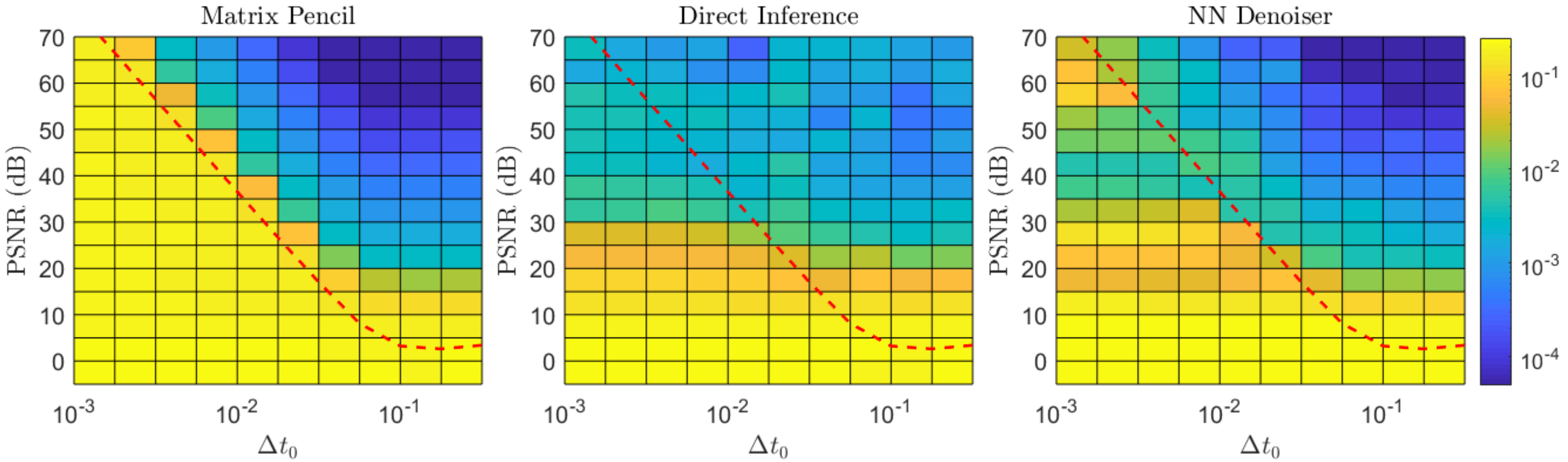
- Fix the first Dirac at $t_0 = 0$ and change $t_1 \in [10^{-0.5}, 10^{-3}]$ evenly on a logarithmic scale with a step of $10^{-0.25}$

- Fixed amplitude $\{\hat{a}_k\}_{k=0}^{1} = 2$ for breakdown PSNR comparison

- Monte Carlo simulations with $I = 10{,}000$ test data for each Δt_0-PSNR pair

Simulation Results

- Both DNN approaches lowers breakdown PSNR
- Denoiser fails to push the breakdown PSNR boundary in high PSNR region



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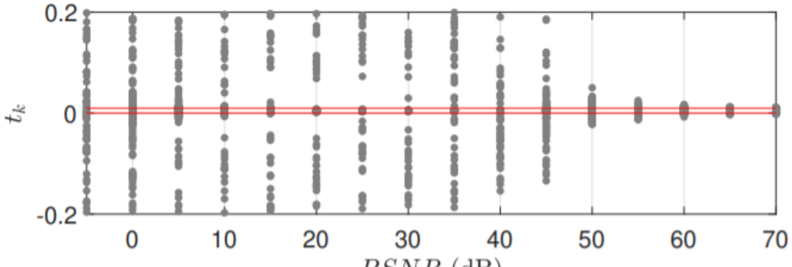
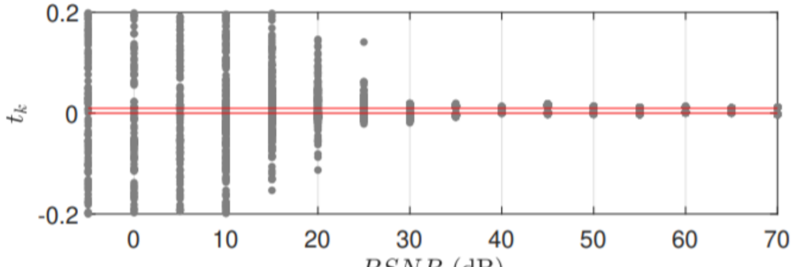
Simulation Results ($\Delta t_0 = 0.01$)

- When the Diracs are close together,
  - Direct inference method using DNN has pushed the breakdown PSNR lower
  - Both methods eventually breaks down when PSNR < 20 dB due to the high noise level



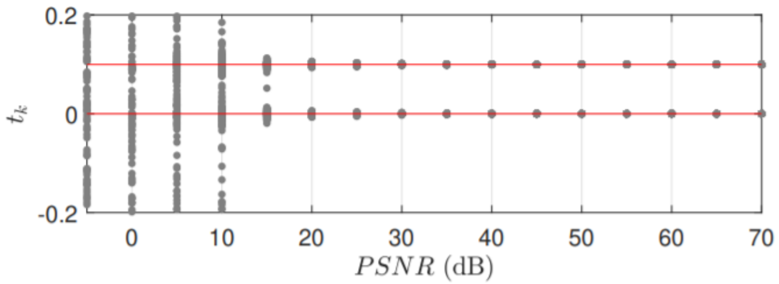
(a) Matrix Pencil



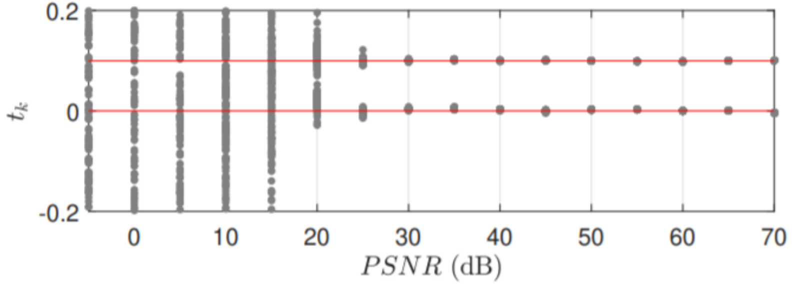
(b) Direct inference

Simulation Results ($\Delta t_0 = 0.1$)

- When the Diracs are sufficiently far apart,
  - The breakdown PSNR is higher for matrix pencil method
  - The centers of the scatters at high PSNR is not entirely aligned with the true locations



(a) Matrix Pencil



(b) Direct Inference

Conclusion and Future Work

- We proposed two DNN-based approaches to retrieve the FRI signal:
  1. Direct inference of FRI parameters
  2. Denoising the samples

- DNN-based methods can reconstruct FRI signals at a low PSNR region where the existing FRI methods would break down, yet with a slight performance compromise in high PSNR region

- Future directions
  - Provide the network with explicit information about the sampling kernel
  - Design network architecture that incorporates the classical methods in an end-to-end training

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Thank You