Multiview Sensing with Unknown Permutations: An Optimal Transport Approach
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**Introduction**

- **Motivation**
  - Traditional security checkpoint requires each person to stand still while being scanned --> slowdowns and long queues
  - By combining the latest developments in optical and depth sensing, tracking, and array processing, we wish to allow scanning of moving individuals with irregular motion as they pass the scanner
  - In this work, we assume an imperfect motion tracking system is available and tackle the image reconstruction task while simultaneously correcting the tracking error

- **Simplified model**
  - Goal: reconstruct \( \mathbf{x} \)
  - As the object \( \mathbf{x} \) moves, it is measured in a sequence of frames. Due to the motion, it undergoes a series of deformations which determines its pose in each frame of the measurements.
  - Motion tracking error is assumed to be corrected by an unknown permutation matrix \( \mathbf{P}_i \)
  - Measurement model for the \( i \)-th view: \( \mathbf{y}_i = \mathbf{A}_i \mathbf{P}_i \mathbf{F} \mathbf{x} + \mathbf{w}_i \)

- **Optimal transport (OT)**
  - A key component of the proposed method
  - Given two probability vectors \( \mathbf{u} \) and \( \mathbf{v} \) and a predefined ground cost matrix \( \mathbf{C} \)
  - Find the optimal coupling \( \mathbf{P} \) between \( \mathbf{u} \) and \( \mathbf{v} \)
  - The ground cost matrix \( \mathbf{C} \) is defined as the sum of all joint probability distributions whose marginals are \( \mathbf{u} \) and \( \mathbf{v} \)
  - When \( M = N \) and \( u = v = \mathbf{1} \), the optimal \( \mathbf{P} \) is a permutation matrix

- **Simulation setup 1**
  - Ground truth pixel value is i.i.d. uniform within each bar
  - Sensing operator \( \mathbf{A} \) has i.i.d. Gaussian entries
  - Number of views is 2
  - Measurement rate is defined as the ratio between total number of measurements (summing over all views) and number of pixels in \( \mathbf{x} \)
  - Increasing number of measurements also increases size of unknown permutation

- **Simulation setup 2**
  - Input SNR is 20dB
  - Fixing number of measurements per view, reconstruction quality improves with increased number of views
  - This can be important for some applications, as the number or measurements can be limited by hardware

**Proposed Method**

- **Assumptions**
  - Support of \( \mathbf{x} \) is known
  - Permutations moving pixels far away from its original location are less likely (i.e., big tracking error is less likely)

- **Optimization formulation**
  \[
  \min_{\mathbf{P}_i} \sum_i |y_i - \mathbf{A}_i \mathbf{P}_i \mathbf{F} \mathbf{x}| + \beta |\delta(\mathbf{P}_i) + \delta(\mathbf{x} - \mathbf{P}_i \mathbf{F} \mathbf{x})| \]

  \( \beta \) determines the pose in each frame of the measurements.

- **Relaxation of equality constraint**
  \[
  \min_{\mathbf{P}_i} \sum_i |y_i - \mathbf{A}_i \mathbf{P}_i \mathbf{F} \mathbf{x}| + \beta |\delta(\mathbf{P}_i) + \delta(\mathbf{x} - \mathbf{P}_i \mathbf{F} \mathbf{x})| \]

  - Equivalent to the relaxed problem if \( u_i = v_i = 1 \)
  - Efficient algorithms from OT literature (e.g., Sinkhorn iterations and its variants) can be applied to solve for \( \mathbf{P}_i \)
  - Other marginals \( \mathbf{u}, \mathbf{v} \) can also be used to further relax the constraint that \( \mathbf{P}_i \) is permutation

- **Connection to OT**
  \[
  \begin{align*}
  \min & \sum_i |y_i - \mathbf{A}_i \mathbf{P}_i \mathbf{F} \mathbf{x}| + \beta \min_{P_i} & (\mathbf{C}(\mathbf{x}, \mathbf{F} \mathbf{x}, \mathbf{P}_i) \mathbf{P}_i) \\
  \text{subject to} & \mathbf{x} = \mathbf{P}_i \mathbf{F} \mathbf{x} \\
  \end{align*}
  \]

  \( \mathbf{C}(\mathbf{x}, \mathbf{F} \mathbf{x}, \mathbf{P}_i) \) is known

- **Proposed algorithm**
  - Alternating between estimation of \( \mathbf{x} \) and \( \mathbf{x}_i \) in (3)
  - Gradient descent for each subproblem
  - Envelope theorem can be used to compute the gradient of minimization function
  - Marginal \( \mathbf{v}_i \) is uniform over support of \( \mathbf{F} \mathbf{x} \) (known)
  - Marginal \( \mathbf{u}_i \) is uniform over estimated support of \( \mathbf{x}_i \)

- **Extensions**
  - Other similarity measure between \( r[\mathbf{n}] \) and \( r[\mathbf{n'}] \), \( \mathbf{x}_i \) and \( \mathbf{P}_i \mathbf{F} \mathbf{x} \), can be used depending on specific problems
  - Regularization for \( \mathbf{x} \) and \( \mathbf{x}_i \) can be easily incorporated

- **Baseline methods for comparison**
  - Since tracking error is small, a naive method is to ignore \( \mathbf{P}_i \)
  - Alternatively, a more straightforward relaxation of the permutation constraint is to replace it with a differentiable penalty

**Summary**

- Signal estimation with unknown permutations is challenging
- In practice, some permutations are more likely than others
- We introduced regularization to promote certain type of permutations
- Further relaxation allowed us making connection to optimal transport, which provides tractable algorithms
- Other regularization for permutations (depending on specific problem) may be translated to choosing certain OT ground cost