Low-Rank and Sparse Decomposition for Joint DOA Estimation and Contaminated Sensors Detection with Sparsely Contaminated Arrays

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- Direction-of-Arrival (DOA) Estimation
- Perfect Array & Sensor Errors

(a) Source localization  
(b) Sonar detection
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- Classical Methods for Sensor Errors
  - Auxiliary sources
  - Perfectly calibrated sensors
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  - Auxiliary sources
  - Perfectly calibrated sensors

- Partly Calibrated Array
  - Number of calibrated sensors
  - Positions of calibrated sensors
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- Classical Methods for Sensor Errors
  - Auxiliary sources
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- Partly Calibrated Array
  - Number of calibrated sensors
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- Sparsely Contaminated Array
  - A few sensors at random positions
  - General case
Contributions:

- Joint DOA estimation and distorted sensors detection
- Problem formulation via low-rank and sparse decomposition (LRSD)
- Problem solved by iteratively reweighted least squares (IRLS)
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Signal Model

Without sensor errors:

\[ x(t) = As(t) + n(t) \quad t = 1, 2, \cdots, T \]

- \( x(t) \in \mathbb{C}^M \): array observation
- \( s(t) \in \mathbb{C}^K \): signal waveform
- \( n(t) \in \mathbb{C}^M \): Gaussian noise
- \( T, M, K \): number of snapshots, sensors, and sources, respectively

With sensor gain and phase errors:

\[ y(t) = \tilde{\Phi}As(t) + n(t) = I\tilde{\Phi}As(t) + n(t) = 1 - 2 - T \]

\( \tilde{\Phi} = I\Phi\Phi = \text{diag} \{ \gamma_1, -\gamma_2, \cdots, -\gamma_M \} \)

\( \gamma_m = 0 \) for perfect sensors

\( \gamma_m \neq 0 \) for contaminated sensors
Proposed Methods

Signal Model

Without sensor errors:

\[ x(t) = As(t) + n(t) \quad t = 1, 2, \cdots, T \]

- \( x(t) \in \mathbb{C}^M \): array observation
- \( A \in \mathbb{C}^{M \times K} \): steering matrix
- \( s(t) \in \mathbb{C}^K \): signal waveform
- \( n(t) \in \mathbb{C}^M \): Gaussian noise
- \( T, M, K \): number of snapshots, sensors, and sources, respectively

With sensor gain and phase errors:

\[ y(t) = \tilde{\Gamma}As(t) + n(t) = (I + \Gamma)As(t) + n(t) \quad t = 1, 2, \cdots, T \]

- \( \tilde{\Gamma} = I + \Gamma \)
- \( \Gamma = \text{diag}\{\gamma\} \)
- \( \gamma = [\gamma_1, \gamma_2, \cdots, \gamma_M]^T \)
- \( \gamma_m \begin{cases} = 0, & \text{for perfect sensors} \\ \neq 0, & \text{for contaminated sensors} \end{cases} \)
Proposed Methods

Problem Formulation via LRSD (1 of 2)

Recall: \( y(t) = (I + \Gamma)A s(t) + n(t) \quad t = 1, 2, \cdots, T \)

Collecting all time-snapshots, matrix-form:

\[
Y = (I + \Gamma)AS + N
\]

- \( Y = [y(1), y(2), \cdots, y(T)] \in \mathbb{C}^{M \times T} \)
- \( S \in \mathbb{C}^{K \times T} \)
- \( N \in \mathbb{C}^{M \times T} \)
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- \( S \in \mathbb{C}^{K \times T} \)
- \( N \in \mathbb{C}^{M \times T} \)

Defining \( Z = AS \) and \( V = \Gamma AS \):

\[
Y = AS + \Gamma AS + N = Z + V + N
\]

- \( Z \in \mathbb{C}^{M \times T} \): of rank \( K \), low-rank matrix
- \( V \in \mathbb{C}^{M \times T} \): row-sparse due to the sparsity of diagonal of \( \Gamma \)
Proposed Methods

*Problem Formulation via LRSD (2 of 2)*

Thanks to the low rank ($Z$) and row-sparse ($V$) structures, propose:

$$\min_{Z, V} ||Y - Z - V||_{F}^{2} + \lambda_{1} ||V||_{2,0} + \lambda_{2} \text{Rank}(Z)$$

- $|| \cdot ||_{F}$: Frobenius norm
- $|| \cdot ||_{2,0}$: $\ell_{2,0}$ mixed norm
- Rank(·): matrix rank
Proposed Methods

Problem Formulation via LRSD (2 of 2)

Thanks to the low rank ($Z$) and row-sparse ($V$) structures, propose:

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- $\| \cdot \|_F$: Frobenius norm
- $\| \cdot \|_{2,0}$: $\ell_{2,0}$ mixed norm
- $\text{Rank}(\cdot)$: matrix rank

Convex relaxation:

$$\min_{Z, V} \| Y - Z - V \|_F^2 + \lambda_1 \| V \|_{2,1} + \lambda_2 \| Z \|_*$$

- $\| \cdot \|_{2,1}$: $\ell_{2,1}$ mixed norm
- $\| \cdot \|_*$: nuclear norm, i.e., sum of singular values
Proposed Methods

Problem Solved by IRLS (1 of 2)

Real-valued form:

\[
\min_{\tilde{Z}, \tilde{V}} \| \tilde{Y} - \tilde{Z} - \tilde{V}\|_F^2 + \lambda_1 \|\tilde{V}\|_{2,1} + \lambda_2 \|\tilde{Z}\|_* 
\]

\[ \tilde{Y} = \begin{bmatrix}
Re\{Y\} & -Im\{Y\} \\
Im\{Y\} & Re\{Y\}
\end{bmatrix} \in \mathbb{R}^{2M \times 2T} \quad \tilde{Z} \in \mathbb{R}^{2M \times 2T} \quad \tilde{V} \in \mathbb{R}^{2M \times 2T} \]
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Real-valued form:

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\]

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\tilde{Z} \in \mathbb{R}^{2M \times 2T} \\
\tilde{V} \in \mathbb{R}^{2M \times 2T}
\]

Handling the non-smoothness:

\[
\min_{\tilde{Z}, \tilde{V}} f = \|\tilde{Y} - \tilde{Z} - \tilde{V}\|_F^2 + \lambda_1 \|[\tilde{V}, \mu 1]\|_{2,1} + \lambda_2 \|[\tilde{Z}, \mu I]\|_*
\]
Derivatives of the objective function:

\[
\frac{\partial f}{\partial \tilde{Z}} = \tilde{Z}(\lambda_2 Q + 2I) + 2(\tilde{V} - \tilde{Y})
\]

\[
\frac{\partial f}{\partial \tilde{V}} = (\lambda_1 P + 2I)\tilde{V} + 2(\tilde{Z} - \tilde{Y})
\]

- \( P = \text{diag}\left(\left(\|\tilde{V}_1\|_2 + \mu^2\right)^{-\frac{1}{2}}, \cdots, \left(\|\tilde{V}_{2M}\|_2 + \mu^2\right)^{-\frac{1}{2}}\right)\)

- \( Q = (\tilde{Z}^T \tilde{Z} + \mu^2 I)^{-\frac{1}{2}} \)
Derivatives of the objective function:

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- \( Q = \left(\tilde{Z}^T\tilde{Z} + \mu^2 I\right)^{-\frac{1}{2}} \)

Setting derivatives to zeros, solutions:

\[
\tilde{Z} = 2(\tilde{Y} - \tilde{V})(\lambda_2 Q + 2I)^{-1} \quad \tilde{V} = 2(\lambda_1 P + 2I)^{-1}(\tilde{Y} - \tilde{Z})
\]
Proposed Methods

*Contaminated Sensors Detection and DOA Estimation*

- The sensors, whose $\ell_2$ norms of their corresponding rows of $\hat{V}$ are far larger than the others, are regarded as contaminated sensors.
Proposed Methods
Contaminated Sensors Detection and DOA Estimation

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- DOA of signals are estimated via the MUSIC spectrum:

$$P(\theta) = \frac{1}{a^H(\theta)(I - U_sU_s^H)a(\theta)}$$

with singular value decomposition $\widehat{Z} = U_s\Sigma_sV_s$
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Setups

- ULA of $M = 10$ sensors
- Distorted sensors: 1st, 4th, 7th, and 8th positions
- $K = 3$ signals with DOAs: \{20°, 50°, 70°\}
- $T = 100$ snapshots
- Regularization parameters: $\lambda_1 = 0.2$, $\lambda_2 = 0.5$, $\mu = 0.1$
Simulation Results

Contaminated Sensors Detection

SNR = -20 dB

SNR = 10 dB

SNR = -10 dB

SNR = 20 dB
Simulation Results

DOA Estimation Performance

![Graph showing DOA estimation performance across different SNR levels for MUSIC with Known Errors, MUSIC without Calibration, LASSO-ADMM, and LRSD-IRLS (Proposed).](image)
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Conclusion:

- Sparsely contaminated array was introduced in DOA estimation.
- We formulated the problem under the framework of LRSD.
- An IRLS technique was derived to solve the resulting problem.
- Numerical results exhibited the effectiveness and superiority in both DOA estimation and contaminated sensors detection.

Outlook:

To guarantee that the proposed method works well, how many sensors at most (with random positions) can be distorted?
Conclusion and Outlook

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- To guarantee that the proposed method works well, how many sensors at most (with random positions) can be distorted?
Thank you for your attention!

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