Low-Rank and Sparse Decomposition for Joint DOA Estimation and Contaminated Sensors Detection with Sparsely Contaminated Arrays

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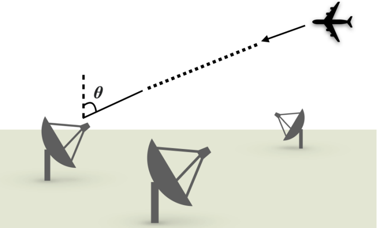
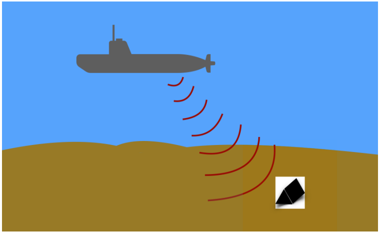
Introduction

- Direction-of-Arrival (DOA) Estimation
- Perfect Array & Sensor Errors



(a) Source localization



(b) Sonar detection

Introduction

- Classical Methods for Sensor Errors
  - Auxiliary sources
  - Perfectly calibrated sensors

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- Partly Calibrated Array
  - Number of calibrated sensors
  - Positions of calibrated sensors

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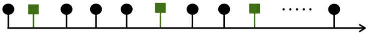
  - Number of calibrated sensors
  - Positions of calibrated sensors

(c) Partly calibrated array

- Sparsely Contaminated Array
  - A few sensors at random positions
  - General case



(d) Sparsely contaminated array

Introduction

Contributions:

- Joint DOA estimation and distorted sensors detection
- Problem formulation via low-rank and sparse decomposition (LRSD)
- Problem solved by iteratively reweighted least squares (IRLS)

Overview

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Without sensor errors:

$$\mathbf{x}(t) = \mathbf{A}\mathbf{s}(t) + \mathbf{n}(t) \quad t = 1, 2, \cdots, T$$

- $\mathbf{x}(t) \in \mathbb{C}^M$: array observation — $\mathbf{A} \in \mathbb{C}^{M \times K}$: steering matrix
- $\mathbf{s}(t) \in \mathbb{C}^K$: signal waveform — $\mathbf{n}(t) \in \mathbb{C}^M$: Gaussian noise
- T, M, K: number of snapshots, sensors, and sources, respectively

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With sensor gain and phase errors:

$$\mathbf{y}(t) = \breve{\mathbf{\Gamma}}\mathbf{A}\mathbf{s}(t) + \mathbf{n}(t) = (\mathbf{I} + \mathbf{\Gamma})\mathbf{A}\mathbf{s}(t) + \mathbf{n}(t) \quad t = 1, 2, \cdots, T$$

- $\breve{\mathbf{\Gamma}} = \mathbf{I} + \mathbf{\Gamma}$
- $\mathbf{\Gamma} = \mathrm{diag}\{\boldsymbol{\gamma}\}$
- $\boldsymbol{\gamma} = [\gamma_1, \gamma_2, \cdots, \gamma_M]^{\mathrm{T}}$
- $\gamma_m \begin{cases} = 0, & \text{for perfect sensors} \\ \neq 0, & \text{for contaminated sensors} \end{cases}$

Recall: $\mathbf{y}(t) = (\mathbf{I} + \mathbf{\Gamma})\mathbf{A}\mathbf{s}(t) + \mathbf{n}(t) \quad t = 1, 2, \cdots, T$

Collecting all time-snapshots, matrix-form:

$$\mathbf{Y} = (\mathbf{I} + \mathbf{\Gamma})\mathbf{A}\mathbf{S} + \mathbf{N}$$

- $\mathbf{Y} = [\mathbf{y}(1), \mathbf{y}(2), \cdots, \mathbf{y}(T)] \in \mathbb{C}^{M\times T} \qquad \mathbf{S} \in \mathbb{C}^{K\times T} \qquad \mathbf{N} \in \mathbb{C}^{M\times T}$

Proposed Methods
Problem Formulation via LRSD (1 of 2)

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Recall: $\mathbf{y}(t) = (\mathbf{I} + \mathbf{\Gamma})\mathbf{A}\mathbf{s}(t) + \mathbf{n}(t) \quad t = 1, 2, \cdots, T$

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Defining $\mathbf{Z} = \mathbf{A}\mathbf{S}$ and $\mathbf{V} = \mathbf{\Gamma}\mathbf{A}\mathbf{S}$:

$$\boxed{\mathbf{Y} = \mathbf{A}\mathbf{S} + \mathbf{\Gamma}\mathbf{A}\mathbf{S} + \mathbf{N} = \mathbf{Z} + \mathbf{V} + \mathbf{N}}$$

- $\mathbf{Z} \in \mathbb{C}^{M\times T}:$ of rank K, low-rank matrix
- $\mathbf{V} \in \mathbb{C}^{M\times T}:$ row-sparse due to the sparsity of diagonal of $\mathbf{\Gamma}$

Thanks to the low rank ($\mathbf{Z}$) and row-sparse ($\mathbf{V}$) structures, propose:

$$\min_{\mathbf{Z},\mathbf{V}} \|\mathbf{Y}-\mathbf{Z}-\mathbf{V}\|_\mathrm{F}^2 + \lambda_1 \|\mathbf{V}\|_{2,0} + \lambda_2 \mathrm{Rank}(\mathbf{Z})$$

- $\|\cdot\|_\mathrm{F}$: Frobenius norm
- $\|\cdot\|_{2,0}$: $\ell_{2,0}$ mixed norm
- $\mathrm{Rank}(\cdot)$: matrix rank

Proposed Methods
Problem Formulation via LRSD (2 of 2)

Thanks to the low rank ($\mathbf{Z}$) and row-sparse ($\mathbf{V}$) structures, propose:

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Convex relaxation:

$$\min_{\mathbf{Z},\mathbf{V}} \|\mathbf{Y}-\mathbf{Z}-\mathbf{V}\|_F^2 + \lambda_1 \|\mathbf{V}\|_{2,1} + \lambda_2 \|\mathbf{Z}\|_*$$

- $\|\cdot\|_{2,1}$: $\ell_{2,1}$ mixed norm
- $\|\cdot\|_*$: nuclear norm, i.e., sum of singular values

Proposed Methods
Problem Solved by IRLS (1 of 2)

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Real-valued form:

$$\min_{\widetilde{\mathbf{Z}},\widetilde{\mathbf{V}}} \|\widetilde{\mathbf{Y}} - \widetilde{\mathbf{Z}} - \widetilde{\mathbf{V}}\|_F^2 + \lambda_1 \|\widetilde{\mathbf{V}}\|_{2,1} + \lambda_2 \|\widetilde{\mathbf{Z}}\|_*$$

- $\widetilde{\mathbf{Y}} = \begin{bmatrix} Re\{\mathbf{Y}\} & -Im\{\mathbf{Y}\} \\ Im\{\mathbf{Y}\} & Re\{\mathbf{Y}\} \end{bmatrix} \in \mathbb{R}^{2M\times 2T} \quad \widetilde{\mathbf{Z}} \in \mathbb{R}^{2M\times 2T} \quad \widetilde{\mathbf{V}} \in \mathbb{R}^{2M\times 2T}$

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Handling the non-smoothness:

$$\min_{\widetilde{\mathbf{Z}},\widetilde{\mathbf{V}}} f = \|\widetilde{\mathbf{Y}} - \widetilde{\mathbf{Z}} - \widetilde{\mathbf{V}}\|_F^2 + \lambda_1 \left\| [\widetilde{\mathbf{V}}, \mu \mathbf{1}] \right\|_{2,1} + \lambda_2 \left\| \begin{bmatrix} \widetilde{\mathbf{Z}} \\ \mu \mathbf{I} \end{bmatrix} \right\|_*$$

Proposed Methods
Problem Solved by IRLS (2 of 2)

Derivatives of the objective function:

$$\frac{\partial f}{\partial \widetilde{\mathbf{Z}}} = \widetilde{\mathbf{Z}}(\lambda_2 \mathbf{Q} + 2\mathbf{I}) + 2(\widetilde{\mathbf{V}} - \widetilde{\mathbf{Y}}) \qquad \frac{\partial f}{\partial \widetilde{\mathbf{V}}} = (\lambda_1 \mathbf{P} + 2\mathbf{I})\widetilde{\mathbf{V}} + 2(\widetilde{\mathbf{Z}} - \widetilde{\mathbf{Y}})$$

- $\mathbf{P} = \operatorname{diag}\left(\left[\left(\|[\widetilde{\mathbf{V}}]_{1,:}\|_2^2 + \mu^2\right)^{-\frac{1}{2}}, \cdots, \left(\|[\widetilde{\mathbf{V}}]_{2M,:}\|_2^2 + \mu^2\right)^{-\frac{1}{2}}\right]\right)$
- $\mathbf{Q} = \left(\widetilde{\mathbf{Z}}^{\mathrm{T}}\widetilde{\mathbf{Z}} + \mu^2\mathbf{I}\right)^{-\frac{1}{2}}$

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- $\mathbf{Q} = \left(\widetilde{\mathbf{Z}}^{\mathrm{T}}\widetilde{\mathbf{Z}} + \mu^2 \mathbf{I}\right)^{-\frac{1}{2}}$

Setting derivatives to zeros, solutions:

$$\widetilde{\mathbf{Z}} = 2(\widetilde{\mathbf{Y}} - \widetilde{\mathbf{V}})(\lambda_2 \mathbf{Q} + 2\mathbf{I})^{-1} \qquad \widetilde{\mathbf{V}} = 2(\lambda_1 \mathbf{P} + 2\mathbf{I})^{-1}(\widetilde{\mathbf{Y}} - \widetilde{\mathbf{Z}})$$

Proposed Methods

Contaminated Sensors Detection and DOA Estimation

- The sensors, whose ℓ_2 norms of their corresponding rows of $\widehat{\mathbf{V}}$ are far larger than the others, are regarded as contaminated sensors.

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- DOA of signals are estimated via the MUSIC spectrum:

$$P(\theta) = \frac{1}{\mathbf{a}^{\mathrm{H}}(\theta)(\mathbf{I} - \mathbf{U}_s\mathbf{U}_s^{\mathrm{H}})\mathbf{a}(\theta)}$$

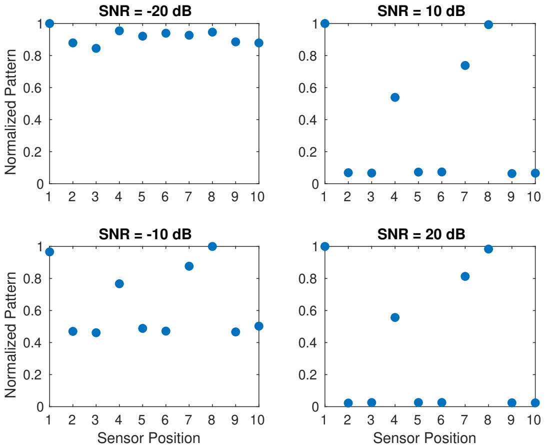
with singular value decomposition $\widehat{\mathbf{Z}} = \mathbf{U}_s \boldsymbol{\Sigma}_s \mathbf{V}_s$

Overview

Simulation Results
Setups

- ULA of $M = 10$ sensors
- Distorted sensors: 1st, 4th, 7th, and 8th positions
- $K = 3$ signals with DOAs: $\{20°, 50°, 70°\}$
- $T = 100$ snapshots
- Regularization parameters: $\lambda_1 = 0.2$, $\lambda_2 = 0.5$, $\mu = 0.1$

Simulation Results
Contaminated Sensors Detection



Simulation Results
DOA Estimation Performance

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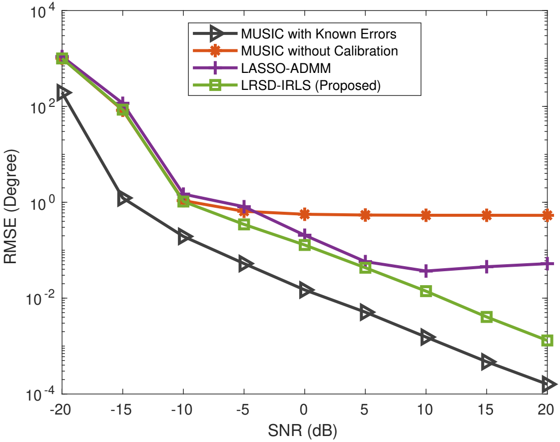

Overview

Conclusion and Outlook

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Conclusion:

- Sparsely contaminated array was introduced in DOA estimation.
- We formulated the problem under the framework of LRSD.
- An IRLS technique was derived to solve the resulting problem.
- Numerical results exhibited the effectiveness and superiority in both DOA estimation and contaminated sensors detection.

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Outlook:

- To guarantee that the proposed method works well, how many sensors at most (with random positions) can be distorted?

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Thank you for your attention!

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