PLUG-IN MEASURE-TRANSFORMED QUASI-LIKELIHOOD RATIO TEST FOR RANDOM SIGNAL DETECTION

Nir Halay and Koby Todros

Dept. of ECE, Ben-Gurion University of the Negev

PROBLEM FORMULATION

Detection of a random signal that lies on a known rank-one subspace:

$$H_0: \begin{cases} \mathbf{X}_n = \mathbf{W}_n, & n = 1, \ldots, N \\ \mathbf{Y}_m = \mathbf{W}^{(s)}_m, & m = 1, \ldots, M \end{cases}$$

$$H_1: \begin{cases} \mathbf{X}_n = S_n \mathbf{a} + \mathbf{W}_n, & n = 1, \ldots, N \\ \mathbf{Y}_m = \mathbf{W}^{(s)}_m, & m = 1, \ldots, M \end{cases}$$

GAUSS-GAUSS DETECTOR

- GLRT detector that assumes jointly Gaussian signal and noise.
- Advantages: simple implementation, ease of performance analysis.
- Disadvantage: sensitive to model mismatch.

PLUG-IN NSDD-GLRT

- Conditional GLRT detector that assumes a compound-Gaussian noise.
- The scatter matrix is replaced by noise-only secondary data ML estimate.
- Advantages: robust against heavy-tailed noise outliers.
- Disadvantage: Computationally demanding in high-dimensions, does not reject large-norm outliers.

MEASURE TRANSFORMED (MT) GQLRT: BASIC IDEA

- Selects a Gaussian probability model that best empirically fits a transformed probability measure of the data.
- By proper choice of the transform the MT-GQLRT can gain enhanced robustness to outliers.
- Have the computational and implementation advantages of the GGD.

PROBABILITY MEASURE TRANSFORM

Let $\mathbf{X} \in \mathbb{C}^p$. Define the measure space $(\mathcal{X}, \mathcal{S}_x, P_{\mathrm{x}})$. Given a non-negative function $u : \mathbb{C}^p \to \mathbb{R}_+$ satisfying $0 < \mathrm{E}[u(\mathbf{X}); P_{\mathrm{x}}] < \infty$. A transform $\mathbb{T}_u : P_{\mathrm{x}} \to Q^{(u)}_{\mathrm{x}}$ is defined as:

$$\mathbb{T}_u[P_{\mathrm{x}}](A) = Q^{(u)}_{\mathrm{x}}(A) \triangleq \int_A \varphi_u(\mathbf{x}) dP_{\mathrm{x}}(\mathbf{x}),$$

where $A \in \mathcal{S}_x$, and

$$\varphi_u(\mathbf{x}) \triangleq \frac{u(\mathbf{x})}{\mathrm{E}[u(\mathbf{X}); P_{\mathrm{x}}]}.$$

The function $u(\cdot)$ is called the MT-function.

THE MEASURE-TRANSFORMED MEAN AND COVARIANCE

- MT-mean: $\boldsymbol{\mu}^{(u)}_{\mathrm{x}} \triangleq \mathrm{E}[\mathbf{X}\varphi_u(\mathbf{X}); P_{\mathrm{x}}]$
- MT-covariance: $\boldsymbol{\Sigma}^{(u)}_{\mathrm{x}} \triangleq \mathrm{E}[\mathbf{X}\mathbf{X}^H \varphi_u(\mathbf{X}); P_{\mathrm{x}}] - \boldsymbol{\mu}^{(u)}_{\mathrm{x}} \boldsymbol{\mu}^{(u)H}_{\mathrm{x}}$
- The mean and covariance under $Q^{(u)}_{\mathrm{x}}$ can be estimated using samples from P_{x}.

$$\hat{\boldsymbol{\mu}}^{(u)}_{\mathrm{x}} \triangleq \sum_{n=1}^{N} \mathbf{X}_n \hat{\varphi}_u(\mathbf{X}_n) \quad \text{and} \quad \hat{\boldsymbol{\Sigma}}^{(u)}_{\mathrm{x}} \triangleq \sum_{n=1}^{N} \mathbf{X}_n \mathbf{X}_n^H \hat{\varphi}_u(\mathbf{X}_n) - \hat{\boldsymbol{\mu}}^{(u)}_{\mathrm{x}} \hat{\boldsymbol{\mu}}^{(u)H}_{\mathrm{x}}$$

where $\hat{\varphi}_u(\mathbf{X}_n) \triangleq u(\mathbf{X}_n) / \sum_{n=1}^{N} u(\mathbf{X}_n)$

ROBUSTNESS TO OUTLIERS

- Define the ϵ-contaminated probability measure:

$$P_\epsilon \triangleq (1-\epsilon) P_{\mathrm{x}} + \epsilon \delta_{\mathbf{y}},$$

where $0 \le \epsilon \le 1$, $\mathbf{y} \in \mathbb{C}^p$, and $\delta_{\mathbf{y}}$ is the Dirac probability measure at $\mathbf{y}$.

- The influence function (IF) [1] of an estimator with statistical functional $\mathrm{H}[\cdot]$:

$$\mathrm{IF}_{\mathrm{H}}(\mathbf{y}; P_{\mathrm{x}}) \triangleq \lim_{\epsilon \to 0} \frac{\mathrm{H}[P_\epsilon] - \mathrm{H}[P_{\mathrm{x}}]}{\epsilon} = \left. \frac{d\mathrm{H}[P_\epsilon]}{d\epsilon} \right|_{\epsilon=0}$$

- Describes the effect on the estimator of an infinitesimal contamination at $\mathbf{y}$.
- An estimator is said to be B-robust if its IF is bounded.

Proposition (Boundedness of the influence function)

The influence functions of the empirical MT-mean and MT-covariance are bounded if the MT-function $u(\mathbf{y})$ and the product $u(\mathbf{y})\|\mathbf{y}\|^2$ are bounded.

MEASURE-TRANSFORMED (MT) GQLRT: DERIVATION OF THE TEST

- Compares the empirical KLDs between $Q^{(u)}_{\mathrm{x}}$ and two normal distributions that are characterized by the MT-mean and MT-covariance under each hypothesis:

$$\begin{aligned} T_u &\triangleq \hat{D}_{\mathrm{KL}}\left[Q^{(u)}_{\mathrm{x}} || \Phi^{(u)}_{\mathrm{x};H_0}\right] - \hat{D}_{\mathrm{KL}}\left[Q^{(u)}_{\mathrm{x}} || \Phi^{(u)}_{\mathrm{x};H_1}\right] \\ &= \left(D_{\mathrm{LD}}\left[\hat{\boldsymbol{\Sigma}}^{(u)}_{\mathrm{x}} || \boldsymbol{\Sigma}^{(u)}_{\mathrm{x};H_0}\right] + \left\| \hat{\boldsymbol{\mu}}^{(u)}_{\mathrm{x}} - \boldsymbol{\mu}^{(u)}_{\mathrm{x};H_0} \right\|^2_{\left(\boldsymbol{\Sigma}^{(u)}_{\mathrm{x};H_0}\right)^{-1}} \right) \\ &- \left(D_{\mathrm{LD}}\left[\hat{\boldsymbol{\Sigma}}^{(u)}_{\mathrm{x}} || \boldsymbol{\Sigma}^{(u)}_{\mathrm{x};H_1}\right] + \left\| \hat{\boldsymbol{\mu}}^{(u)}_{\mathrm{x}} - \boldsymbol{\mu}^{(u)}_{\mathrm{x};H_1} \right\|^2_{\left(\boldsymbol{\Sigma}^{(u)}_{\mathrm{x};H_1}\right)^{-1}} \right) \underset{H_0}{\overset{H_1}{\gtrless}} \tau \end{aligned}$$

- Equivalent test-statistic under any MT-function $u(\mathbf{x}) = \upsilon(\mathbf{P}^{\perp}_{\mathbf{a}} \mathbf{x})$, $\upsilon : \mathbb{C}^p \to \mathbb{R}_+$

$$T'_u = \frac{\mathbf{a}^H \left(\boldsymbol{\Sigma}^{(u)}_{\mathbf{W}}\right)^{-1} \hat{\mathbf{C}}^{(u)}_{\mathrm{x}} \left(\boldsymbol{\Sigma}^{(u)}_{\mathbf{W}}\right)^{-1} \mathbf{a}}{\mathbf{a}^H \left(\boldsymbol{\Sigma}^{(u)}_{\mathbf{W}}\right)^{-1} \mathbf{a}},$$

where $\hat{\mathbf{C}}^{(u)}_{\mathrm{x}} \triangleq \hat{\boldsymbol{\Sigma}}^{(u)}_{\mathrm{x}} + \hat{\boldsymbol{\mu}}^{(u)}_{\mathrm{x}} \hat{\boldsymbol{\mu}}^{(u)H}_{\mathrm{x}}$.

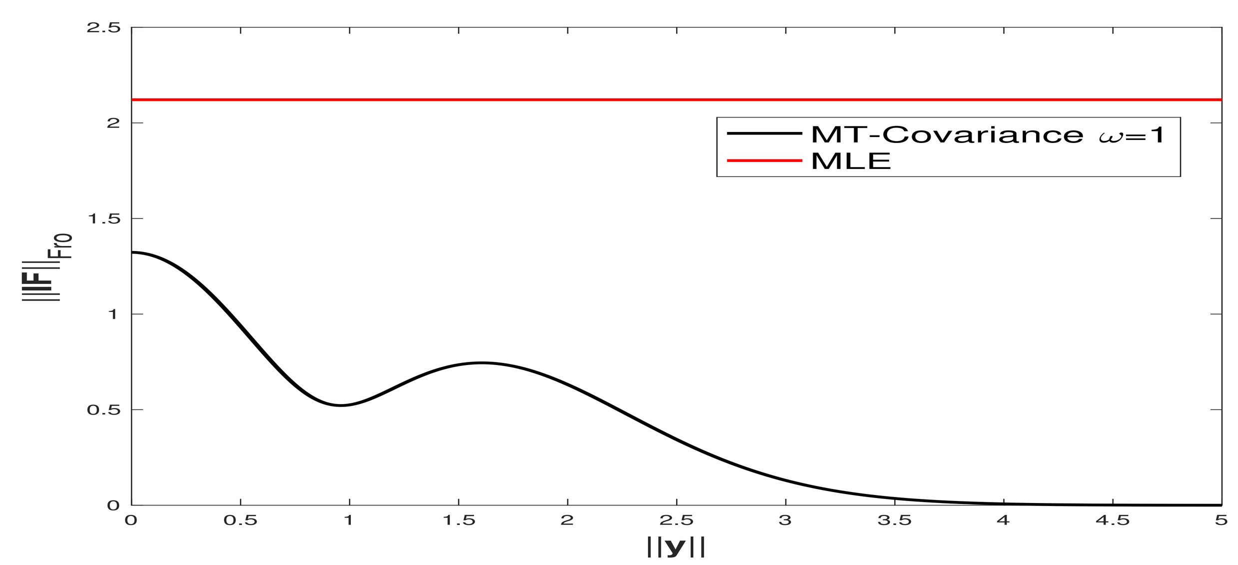
PLUG-IN MEASURE-TRANSFORMED GQLRT

- Replace $\boldsymbol{\Sigma}^{(u)}_{\mathbf{W}}$ by its empirical estimate obtained from the secondary data:

$$T''_u \triangleq \frac{\mathbf{a}^H \left(\hat{\boldsymbol{\Sigma}}^{(u)}_{\mathbf{Y}}\right)^{-1} \hat{\mathbf{C}}^{(u)}_{\mathrm{x}} \left(\hat{\boldsymbol{\Sigma}}^{(u)}_{\mathbf{Y}}\right)^{-1} \mathbf{a}}{\mathbf{a}^H \left(\hat{\boldsymbol{\Sigma}}^{(u)}_{\mathbf{Y}}\right)^{-1} \mathbf{a}} \underset{H_0}{\overset{H_1}{\gtrless}} t,$$

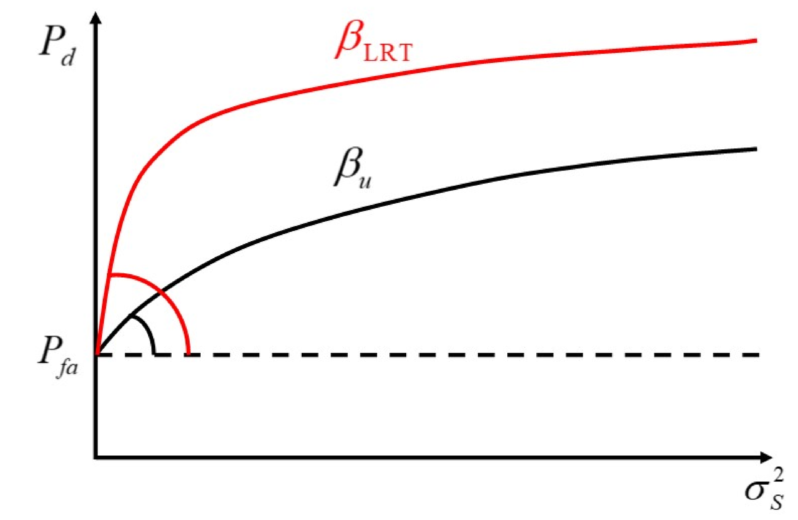
- Under some mild regularity conditions T''_u is asymptotically normal.
- To induce outlier resilience, choose the Gaussian MT-function:

$$u_G(\mathbf{x}; \omega) = \exp\left(-\left\|\mathbf{P}^{\perp}_{\mathbf{a}} \mathbf{x}\right\|^2 / \omega^2\right), \ \omega \in \mathbb{R}_{++}.$$



SELECTION OF THE GAUSSIAN MT-FUNCTION WIDTH PARAMETER

- **Principle:** control the asymptotic local power sensitivity to change in the signal variance relative to the omniscient LRT under Gaussian data.



- **Selection Rule:**

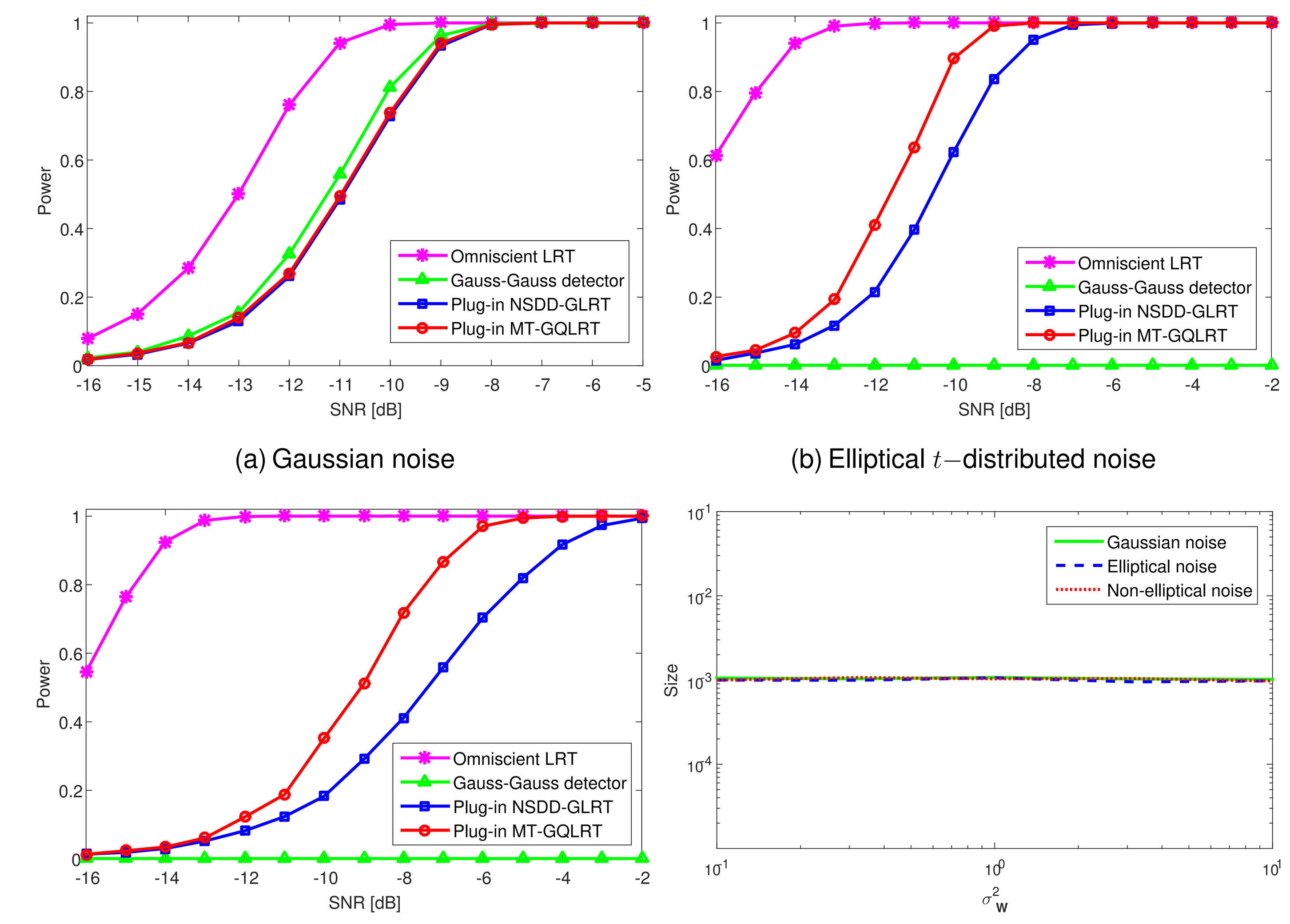
$$\omega^*_{\mathrm{Y}} = \inf\left\{\omega \in \mathbb{R}_+ : \hat{R}_{\mathrm{Y}}(\omega) = r\right\}.$$

$$\hat{R}_{\mathrm{Y}}(\omega) \triangleq \left. \frac{\partial \hat{\beta}_{u_G}}{\partial \sigma^2_S} \Big/ \frac{\partial \hat{\beta}_{LRT}}{\partial \sigma^2_S} \right|_{\sigma^2_S = 0} = \frac{\sqrt{N \max\left(0, \det\left(\mathbf{I}_p - \frac{1}{\omega^4}\left(\mathbf{P}^{\perp}_{\mathbf{a}} \hat{\boldsymbol{\Sigma}}^{(u_G)}_{\mathrm{Y}}(\omega)\right)^2\right)\right)} + Q^{-1}(\alpha)}{\sqrt{N} + Q^{-1}(\alpha)}$$

- **Constant false alarm rate** (CFAR) w.r.t. the noise power.

EXAMPLES

- Parameters: 8-element ULA, $N = M = 500$, $P_{fa} = 10^{-3}$, $r = 0.9$, BPSK signal.



(a) Gaussian noise (b) Elliptical t−distributed noise

(c) Non-elliptical noise (d) CFAR analysis

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