Sparsity Driven Latent Space Sampling for Generative Prior based Compressive Sensing

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Compressive Sensing:

Linear inverse problem with the measurement model:

\[ y = Ax, \]  
\[ y \in \mathbb{R}^m, \quad x \in \mathbb{R}^n, \quad A \in \mathbb{R}^{m \times n}, \quad m \ll n. \]  

Problem 0:

\[ P_0 : \min_x \| x \|_0, \quad \text{s.t.} \quad y = Ax \]
Recover $\mathbf{x} \in \mathbb{R}^n$ from $\mathbf{y} \in \mathbb{R}^m$, such that $m \ll n$. The measurement model is given as:

$$\mathbf{y} = A\mathbf{x}, \quad \mathbf{x} = G_\theta(\mathbf{z}),$$

(3)

where $A \in \mathbb{R}^{m \times n}$ satisfies Set - Restricted Eigenvalue Condition (S-REC), $G_\theta$ is the generator model with latent input $\mathbf{z} \in \mathbb{R}^k$ and parameter $\theta$.

**Definition: S-REC**

[Bora et al., ICML, 2017] Let $\mathcal{S} \subseteq \mathbb{R}^n$. For some parameters $\gamma > 0, \delta \geq 0$, a matrix $A \in \mathbb{R}^{m \times n}$ is said to satisfy the $S$-REC($\mathcal{S}, \gamma, \delta$) if $\forall \mathbf{x}_1, \mathbf{x}_2 \in \mathcal{S}$,

$$\|A(\mathbf{x}_1 - \mathbf{x}_2)\| \geq \gamma \|\mathbf{x}_1 - \mathbf{x}_2\| - \delta$$

(4)
Motivation

- Natural images do not lie on a single connected manifold, but rather on a union-of-submanifolds.

- A single generator cannot correctly model a distribution that lies on disconnected manifolds [Khayatkhoei et al., NeuRIPS, 2018].

- Disconnected latent space model is considered for data lying on a disconnected manifold.
Contribution

- We propose a union-of-submanifolds to model the true data distribution.

- An optimization framework, namely, proximal meta-learning (PML) algorithm to promote sparsity into the latent variable.

- Sample complexity bounds for the proposed union-of-submanifolds model.

- At higher compression ratio, the performance of SDLSS is superior to state-of-art deep compressed sensing (DCS) [Wu et al., ICML, 2019]
Union-of-Submanifolds

- The latent variable $\mathbf{z} \in \mathbb{R}^k$ is assumed to $s$-sparse.

- Sparsity assumption on the input latent space $\mathcal{Z}$ divides the latent space into $\binom{k}{s}$ subspaces $\mathcal{W}_i$.

- The generator model $G_\theta$ transforms each subspace $\mathcal{W}_i$ to sub-manifold $S_i$. 
Union-of-Submanifolds

The domain of reconstruction is a union-of-submanifolds:

$$S_{s,G_\theta} = \bigcup_i S_i,$$

(5)

where $S_i$ is the submanifold generated by $G_\theta(z)$, with $\|z\|_0 \leq s$. 

Figure: Union-of-Submanifolds
Sparsity Driven Latent Space Sampling (SDLSS)

**Measurement model:**

\[
y = Ax \quad \text{s.t.} \quad x = G_\theta(z), \quad \|z\|_0 \leq s. \quad (6)
\]

\(x\) is assumed to lie in a union of sub-manifolds \(S_{s,G_\theta}\).

**Optimization problem:**

\[
\min_{z,\theta} \|z\|_0 \quad \text{s.t.} \quad \|y - AG_\theta(z)\|_2 \leq \epsilon, \quad (7)
\]

with \(A\) satisfying set-restricted eigenvalue condition (S-REC) and \(z \in \mathbb{R}^k\) is the latent space with sparsity at most \(s\).
Sparsity Driven Latent Space Samping (SDLSS)

Optimization cost:

\[
\min_{z, \theta} \left( L_G + L_A \right),
\]

where

\[
L_G = \mathbb{E}_z \{ E_\theta(y, z) = \| y - AG_\theta(z) \|_2^2 + \| z \|_0 \}
\]

\[
L_A = \mathbb{E}_{x_1, x_2} \{ (\| Ax_1 - x_2 \|_2 + \delta - \gamma \| x_1 - x_2 \|_2)^2 \}
\]

\( L_G \) and \( L_A \) represent the measurement loss and \( S - REC \) loss, respectively.
Enforcing Sparsity

The proximal step is introduced to enforce sparsity in the latent space via meta-learning [Finn et al., ICML, 2017].

\[ E_\theta(y_i, z_i) = \|y_i - AG_\theta(z_i)\|^2_2 + \|z_i\|_0, \]  

\[ \hat{z}_i = \arg \min_{z_i} E_\theta(y_i, z_i), \]  

\[ = Ps(z_i - \beta \nabla_z f(y_i, z_i)), \]

where \( f(y_i, z_i) = \|y_i - AG_\theta(z_i)\|^2_2 \), \( \beta \) is the learning rate, and \( Ps(u) \) is the hard-thresholding operator that sets all but the largest (in magnitude) \( s \) elements of \( u \) to 0.
Algorithm 1: Sparsity Driven Latent Sampling (SDLSS) for Generative Prior

**Input:** data = \{x_i\}_{i=1}^{N}, sensing matrix A, generator G_\theta, learning rate \alpha, number of latent optimization steps T, measurement error threshold \epsilon and sparsity factor s.

repeat
  Initialize generator network parameter \theta.
  for \( i = 1 \) to \( N \) do
    Measure the signal \( y_i = Ax_i \)
    Sample \( z \sim N(0, I) \)
    for \( t = 1 \) to \( T \) do
      \( \hat{z}_i = P_s(z_i - \beta \nabla_z f(y_i, z_i)) \)
    end for
  end for
  \( \mathcal{L} = \mathcal{L}_G + \mathcal{L}_A \)
  Update \( \theta \leftarrow \theta - \alpha \frac{\partial \mathcal{L}}{\partial \theta} \)
  until \( \| y - AG_\theta(\hat{z}) \|_2 \leq \epsilon \)
Theoretical Results

Theorem

Let $G_\theta : \mathbb{R}^k \to \mathbb{R}^n$ be a generative model with $d$-layers with ReLU activation and at most $h$ neurons in each layer. Let $A \in \mathbb{R}^{m \times n}$ be a random Gaussian matrix with IID entries such that $A_{i,j} \sim \mathcal{N}(0, \frac{1}{m})$ and satisfies $S$-REC($S_{s,G_\theta}, (1 - \alpha), 0$). For any $x \in \mathbb{R}^n$, if the number of measurements given by $y = Ax \in \mathbb{R}^m$ is $O(sd \log \frac{kh}{s})$, where $s$ is sparsity of the latent variable $z$, then with probability $1 - \exp(-\Omega(m))$:

$$
\|G_\theta(\hat{z}) - x\|_2 \leq C \min_{z \in \mathbb{R}^k, \|z\|_0 \leq s} \|G_\theta(z) - x\|_2 + 2\epsilon,
$$

(10)

where $\hat{z}$ minimizes $\|y - AG_\theta(z)\|_2^2$ to within additive $\epsilon$ of the optimum, $C$ is a constant, $\Omega$ is the asymptotic lower bound, and $0 < \alpha < 1$. 
MNIST: Effect of Sparsity

Reconstruction error on test data as a function of sparsity; $m = 10$ and latent dimension $k = 784$. 

![Graph showing reconstruction error (RE) in dB as a function of sparsity (s) for MNIST dataset. The graph indicates a decrease in RE with increasing sparsity up to a certain point, after which RE starts to increase again.]
PSNR evaluation of SDLSS and DCS as a function of measurements $m$ for the latent dimension $k = 100$ and sparsity $s = 80$. 

Graph showing the PSNR in dB for SDLSS and DCS as a function of measurements $m$.
Results: MNIST

Ground truth

DCS
PSNR = 15.4 dB

SDLSS
PSNR = 15.6 dB

- Deep Compressed Sensing (DCS) [Wu et al., ICML, 2019]
Results: MNIST

DCS

SDLSS
A. Bora, A. Jalal, E. Price, and A. G. Dimakis
Compressed Sensing using Generative Models

V. Shah, and C. Hegde
Solving Linear Inverse Problems using GAN Priors: An Algorithm with Provable Guarantees

P. Hand, O. Leong, and V. Voroninski
Global Guarantees for Enforcing Deep Generative Priors by Empirical Risk

Y. Wu, M. Rosca, and T. Lillicrap
Deep Compressed Sensing

C. Finn, P. Abbeel, and S. Levine
Model-Agnostic Meta-Learning for Fast Adaptation of Deep Networks

M. Khayatkhoei, M.K. Singh, and A. Elgammal
Disconnected Manifold Learning for Generative Adversarial Networks
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