SUPER-RESOLUTION DOA ESTIMATION FOR ARBITRARY ARRAY GEOMETRIES USING A SINGLE NOISY SNAPSHOT

ICASSP 2019 Presentation

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May 16, 2019
Outline

1. Introduction and Notation
2. Details of Proposed Method
3. Simulations
4. Conclusion
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1. Introduction and Notation
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DOA Estimation Methods

Classical Methods
- Non adaptive: Conventional delay-sum beamformer (CBF)
- Data adaptive: MVDR, MUSIC, ESPRIT

Compressed Sensing (CS) based Sparse Methods
- On-grid sparse DOA estimation - has offgrid (discretization) problem
- Off-grid DOA methods
  - Fixed grid
  - Dynamic grid
- **Gridless method using super-resolution (SR) theory** [Candès and Fernandez-Granda 2014] **for arrays** [Xenaki and Gerstoft 2015]
  - Based on atomic norm or total variation (TV) norm
  - Uses convex optimization (LMI and SDP)
  - Only applicable for ULAs [Xenaki and Gerstoft 2015]
Objective of Proposed Research

- Develop search-free gridless super-resolution DOA method
  - To eliminate offgrid problem of CS
- Extend method to arbitrary array geometries
  - Non-uniform arrays
  - Random Planar 2-D arrays
  - Circular arrays
- Applicable for coherent sources, single snapshot case
Data Model for Arbitrary Array DOA Estimation

- $M$ sensors, known sensor positions
- $L$ sources with unknown azimuth DOAs $\theta = \{\theta_1, \theta_2 \ldots, \theta_L\}$
- Assumptions:
  - far-field, narrow band sources
  - unknown source amplitudes
  - unknown number of sources ($L$)
- Objective is to estimate $\theta$, given the data at the sensors
- Array snapshot vector

$$y(t) = A(\theta)s(t) + n(t) \in \mathbb{C}^{M \times 1}$$

$s(t) = [s_1(t), s_2(t), \ldots, s_L(t)]^T \in \mathbb{C}^{L \times 1}$ is source amplitude vector

$n(t) = [n_1(t), n_2(t), \ldots, n_M(t)]^T \in \mathbb{C}^{M \times 1}$ is noise vector
Array manifold $A(\theta) \triangleq [a(\theta_1), \ldots, a(\theta_L)] \in \mathbb{C}^{M \times L}$

Steering vector $a(\theta_l)$ for the $l$-th source from direction $\theta_l$

$$a_m(\theta_l) = e^{-j2\pi f \tau_m(\theta_l)} = e^{-j(2\pi/\lambda)u_{\theta_l}^T p_m}$$

$f, \lambda$: frequency, wavelength

$\tau_m(\theta_l) = u_{\theta_l}^T p_m / v$

delay at $m$-th sensor for $l$-th source

$p_m$: position vector of $m$-th sensor

$u_{\theta_l}$: unit vector in source direction $\theta_l$

$v$: wave propagation speed
Sparse DOA Estimation: Discrete vs Continuous

- DOA estimation as a sparse signal reconstruction problem

\[
\min_{x \in \mathbb{C}^K} \|x\|_1 \quad \text{s.t.} \quad \|y - A(\theta_D)x\|_2 \leq \epsilon
\]

\(x\) sparse

- \(A(\theta_D) \in \mathbb{C}^{M \times K}\): dictionary of steering vectors
- \(\theta_D = \{\theta : \theta = -\pi + 2\pi k/K, \; k = 1, \ldots, K\}\): discrete grid of angles

- Sparse DOA estimation over continuous domain

\[
\min_{x} \|x\|_A \quad \text{s.t.} \quad \|y - Sx\|_2 \leq \delta
\]

\[x(\theta) = \sum_{l=1}^{L} s_l \delta(\theta - \theta_l), \quad Sx = \int_{-\pi}^{\pi} a_m(\theta)x(\theta)d\theta, \quad m = 1, \ldots, M\]
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Data Model in Continuous Angle Domain

- Source amplitude function in continuous angle domain

\[ x(\theta) = \sum_{l=1}^{L} s_l \delta(\theta - \theta_l), \quad \text{with atomic norm } \| x \|_A = \sum_{l=1}^{L} |s_l| \]

- Array snapshot vector

\[ y = Sx + n, \quad \text{where } y_m = n_m + \int_{-\pi}^{\pi} a_m(\theta)x(\theta)d\theta, \quad m = 1, \ldots, M \]

\( S(\theta) \) is the array manifold surface with \( m \)-th component \( a_m(\theta) \)

\[ a_m(\theta) = e^{-j(2\pi/\lambda)u_0^T\mathbf{p}_m} = \exp\{-j2\pi(|\mathbf{p}_m|/\lambda)\cos(\theta - \angle\mathbf{p}_m)\} \]
### Proposed Method: Primal and Dual Problems

#### Primal Problem

\[
\min_x \|x\|_A \quad \text{s.t.} \quad \|y - Sx\|_2 \leq \delta
\]

#### Dual Problem

\[
\max_{c \in \mathbb{C}^M} \Re\{c^H y\} - \delta\|c\|_2 \quad \text{s.t.} \quad \|S(\theta)^H c\|_\infty \leq 1
\]

- \(b(\theta) = S(\theta)^H c = \sum_{m=1}^{M} a^*_m(\theta)c_m\)
  - \(c\) is a vector of Lagrange multipliers (dual variables)
  - \(|b(\theta)| = 1\) for true source directions
- For ULA, \(b(\theta)\) is a polynomial in \(z = e^{-j(2\pi/\lambda)d \sin \theta}\)

\[
S(\theta)^H c = \sum_{m=1}^{M} c_m e^{-j(m-1)(2\pi/\lambda)d \sin \theta} = \sum_{m=1}^{M} c_m z^{(m-1)}
\]
For arbitrary arrays, $b(\theta)$ does not have a direct polynomial form.

Fourier Domain approach, motivated by [Rübsamen and Gershman 2009] also [Doron & Doron, 1994]

- $b(\theta) = S(\theta)^H c = \sum_{m=1}^M a_m^*(\theta) c_m$

- $a_m^*(\theta)$ periodic $\Rightarrow$ $b(\theta)$ periodic $\Rightarrow$ Fourier Series (FS)

- $b(\theta) = \sum_{k=-\infty}^{\infty} B_k e^{jk\theta}$, where $B_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} b(\theta) e^{-jk\theta} d\theta$

- $a_m^*(\theta)$ is smooth, bandlimited $\Rightarrow$ $b(\theta)$ is bandlimited

- **Finite** Fourier Series ($2N + 1$ coeffs) $\quad b(\theta) = \sum_{k=-N}^{N} B_k e^{jk\theta}$

- $b(\theta) \rightarrow b(z) \bigg|_{z=e^{j\theta}}$ is the **dual polynomial**
Fourier Domain Representation of $a_m(\theta)$

- How to get $\hat{B}_k$'s? $B_k = \sum_{m=1}^{M} \alpha_m[k] c_m$

- $\alpha_m[k]$ are FS coeffs of $a^*_m(\theta) = \exp\{j2\pi(|p_m|/\lambda)\cos(\theta - \angle p_m)\}$

- **DFT** is used to obtain finite FS of a bandlimited function
  - Compute $\hat{\alpha}_m[k]$ via $P$-point DFT; $P = 2N + 1$, $\Delta \theta = 2\pi/P$

  approximation $\hat{\alpha}_m[k] \approx \alpha_m[k]$

  $$\hat{\alpha}_m[k] = \frac{1}{P} \sum_{l=-N}^{N} a^*_m(l\Delta \theta)e^{-j(2\pi/P)lk}$$

- Now, $\hat{B}_k = \sum_{m=1}^{M} \hat{\alpha}_m[k] c_m$, so we have $b(\theta) \approx \sum_{k=-N}^{N} \hat{B}_k e^{jk\theta} \to \hat{b}(z)$,

  $$\begin{bmatrix} \hat{B}_{-N} & \hat{B}_{-(N-1)} & \ldots & \hat{B}_N \end{bmatrix}^T \triangleq h = G^H c,$$

  $G^H = [\hat{\alpha}_m[k]]_{P \times M}$; $m$-th column has FS coefficients of $a^*_m(\theta)$
Fourier Domain Bandwidth Approximation of $a_m(\theta)$

- Selection of $P$ for accurate polynomial representation
  - FS bandwidth of $a_m(\theta) = \exp\left\{-j2\pi(|p_m|/\lambda)\cos(\theta - \angle p_m)\right\}$
  - Plot magnitude of $\hat{\alpha}_m[k]$ vs. $|p|/\lambda$

(a) DFT spectrum of $a^*_m(\theta)$ (20 log_{10}|\alpha_k| dB) as a function of $k$ and $|p|/\lambda$,
(b) $P$ vs. normalized distance $|p|/\lambda$ for different spectral cutoff levels ($\gamma$).

- Linear rule for $P$ w.r.t distance $|p|$ of farthest sensor from reference

For $\gamma = -160$ dB, $P = 15.9|p|/\lambda + 27.03$
Semidefinite Programming and Source Recovery

- Dual Program to Semidefinite Program (SDP)

\[
\begin{align*}
\max_{c, H} & \Re \{c^H y\} - \delta \|c\|_2; \quad \text{s.t.} \quad \begin{bmatrix} H_{P \times P} & G_{P \times M}^H \end{bmatrix} \begin{bmatrix} c_{M \times 1} \end{bmatrix} \succeq 0, \\
\sum_{i=1}^{P-j} H_{i,i+j} & = \begin{cases} 1, & j = 0 \\
0, & j = 1, \ldots, P - 1. \end{cases}
\end{align*}
\]

SDP has \(n = P^2/2 + M\) variables. Worst case complexity \(O(n^3)\)

- Recover source DOAs \(\hat{\theta}\) from unit-circle roots of nonnegative poly.

\[
p(z) = 1 - |\hat{b}(z)|^2 = \sum_{k=-(P-1)}^{P-1} r_k z^k
\]

\(r_k = \sum_j h_j h^*_j \) are autocorrelation coeffs of \(h_* = G^H c_*\)

- Recover source amplitudes by least-squares

\[
\hat{s} = A(\hat{\theta})^\dagger y
\]
Algorithm: Super-Resolution DOA for Arbitrary Array

**Input:** Array snapshot vector $y \in \mathbb{C}^M$, wavelength $\lambda$, number of Fourier coeffs $P$

1. For the sensor positions, compute $G^H = [\hat{\alpha}_m[k]]_{P \times M}$ using the DFT to obtain the FS of the array manifold (OFF-LINE)
2. Estimate noise level, and then set $\delta$
3. Using $G^H$ and $y$ as inputs, solve the SDP to find optimal $c_*$
4. Compute the optimal dual polynomial coefficients-vector $h_*$, using $h_* = G^H c_*$
5. Estimate DOAs $\hat{\theta}$ by finding the unit-circle roots of nonnegative polynomial $p(z)$
6. Eliminate extraneous zeros via $\ell_1$ recovery
7. Recover source amplitudes $\hat{s}$ by least squares
Running Time Examples

- The observed time complexity seems to grow as $P^2$
- SDP has $n = P^2/2 + M$ variables.
- Significantly less than the worst case complexity of $O(n^3)$

<table>
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<th>Case</th>
<th>P</th>
<th>Radius</th>
<th>Time for SDP</th>
<th>Poly. rooting</th>
<th># Iterations</th>
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<td>0.04 sec</td>
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<td>$9.75\lambda$</td>
<td>57.9 sec</td>
<td>0.37 sec</td>
<td>19</td>
</tr>
</tbody>
</table>

Intel core i7 processor, $M = 40$, Three sources
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Simulations

- Simulations for Uniform Circular and Random Planar Arrays (Noise-free)

- Performance Evaluation using Success Probability (Noise-free)

- Simulations for Noisy Case
  - White and Colored Noise Examples
  - $\ell_1$ Recovery Result
  - Performance Evaluation Vs. Signal to Noise Ratio (SNR)

  \[
  \text{SNR} = \frac{\text{Source Power}}{\text{Noise Power}} \quad \text{at each sensor}
  \]

- All Simulations use Coherent Sources and Single Snapshot
Simulation for Uniform Circular Array (UCA)

(a) Dual Polynomial

(b) Nonnegative Polynomial

(c) Zeros of $p(z)$

(d) CBF vs. Proposed

UCA with $r = 2\lambda$, $M = 40$, $P = 61$. Sources at $-10.3^\circ$, $30.5^\circ$, $70.7^\circ$, magnitudes $5, 30, 7$. Noise-free case: Perfect Estimates for DOAs and Mags
Simulation for Random Planar Array (RPA)

(a) Random Planar Array (RPA)

(b) CBF vs. Proposed Method

Result for RPA with $M = 30$, $P = 61$. Farthest sensor at $r/\lambda \approx 2$.

Three sources at DOAs $-65.1^\circ$, $37.5^\circ$, $50.7^\circ$, equal magnitudes.

Noise-free case: estimates of directions and magnitudes are perfect.
Performance Evaluation for Resolution

Success probability of \( M = 40 \) UCA (a) versus \( r/\lambda \) and \( P \), with fixed \( \Delta_{\text{min}} = 10^\circ \).

(b) versus minimum source separation \( \Delta_{\text{min}} \) and \( L \) with fixed \( r/\lambda = 1.59 \).

- **Success probability**
  - Fig. (a): 50 random trials for each \( P \) and \( r/\lambda \). Fixed \( \Delta_{\text{min}} = 10^\circ \)
    - \( L = 10 \) sources with random DOAs \( \sim \mathcal{U}(-\pi, \pi] \)
    - Success declared when all DOAs are estimated within 0.001° error
  - Fig. (b): Fixed radius \( r/\lambda = 1.59 \), \( P = 53 \), and 10 trials
Simulations for Noisy Case: Colored Noise Example

(a) noise spectrum \((1/f)\)

(b) Zeros of \(p(z)\)

(c) CBF vs. Proposed Method

UCA with \(r = 2\lambda, M = 40\) sensors, \(P = 63\). Two sources at \(40^\circ, 50^\circ\); SNR = 20 dB.
Simulations for Noisy Case: RPA, $M = 30$

Result for RPA with $M = 30$, $P = 63$, $\max |p| \approx 2\lambda$.

Two equal magnitude sources at $60^\circ$ and $70^\circ$.

SNR = 20 dB. $\delta = 1.4e_n$.

Minimum sensor spacing = $\lambda/4$. 

DOA RMSE = $0.8882^\circ$
Amplitude RMSE = $0.4693$
Simulations for Noisy Case: RPA, $M = 40$

Result for RPA with $M = 40$, $P = 63$, max $|p| \approx 2\lambda$.

Two equal magnitude sources at $60^\circ$ and $70^\circ$.

SNR = 20 dB. $\delta = 1.4e_n$.

Minimum sensor spacing = $\lambda/4$.  

DOA RMSE = 0.5583°  
Amplitude RMSE = 0.3652
Simulations for Noisy Case: $\ell_1$ Recovery

Result for UCA with \( \text{radius} = 2\lambda, \ M = 40, \ P = 63. \ \delta = 1.4e_n. \)
Five sources with \( \text{SNR} = 5 \text{dB} \) at \(-10.7^\circ, 27.5^\circ, 40^\circ, 73.7^\circ \) and \(-151.1^\circ\)

- Extraneous roots from polynomial rooting
  - Need $\ell_1$ recovery to remove unwanted roots
- Estimate amplitudes by least-squares

DOA RMSE = 0.5617°
Amplitude RMSE = 0.2016
Performance Evaluation vs. SNR

(a) Source Separation = 10°

(b) Source Separation = 30°

DOA accuracy vs. SNR for UCA with $r = 2\lambda$, $M = 30$, and $P = 63$.

50 trials, two sources at random DOAs in each trial.

Additive noise $\mathcal{CN}(0, \sigma)$ per sensor $\Rightarrow e_n = \mathbb{E}[\|n\|_2] = \sqrt{M}\sigma^2$
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Conclusion

- Search-free gridless SR DOA method for arbitrary arrays using single noisy snapshot
  - Formulated problem as an atomic norm minimization
  - Fourier domain approach for polynomial representation of manifold
  - Finite SDP formulation for arbitrary arrays, solvable in polynomial time

- No strong source masking weak source problem, unlike CBF

- Applicable for coherent sources, single snapshot, and colored or white noise scenarios

- Larger impact: Applicable to generic data model involving periodic measurement functions, and to other applications.
Thank You!