MODIFIED ARCSINE LAW FOR ONE-BIT SAMPLED STATIONARY SIGNALS WITH TIME-VARYING THRESHOLDS

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1. Motivation

•$\mathbf{x}$: A zero-mean stationary Gaussian input signal.
•$R_{\mathbf{x}}(l)$: The autocorrelation function of $\mathbf{x}$ for the lag l.
•$\boldsymbol{\tau}$: A threshold which is considered zero in previous works.
•$\mathbf{y}$: The output process (one-bit data).
•$R_{\mathbf{y}}(l)$: The autocorrelation function of $\mathbf{y}$ for the lag l.

The autocorrelation function of $\mathbf{y}$ in lag l is connected to that of $\mathbf{x}$ via the arcsine law:

$$R_{\mathbf{y}}(l) = \frac{2}{\pi}\sin^{-1}\left(\frac{R_{\mathbf{x}}(l)}{R_{\mathbf{x}}(0)}\right). \quad (1)$$

Arcsine law has two main drawbacks. It considers only the zero threshold which causes information loss due to normalized autocorrelation recovery. In this paper, our goal is to modify the arcsine law by considering the time-varying threshold which enables us to recover the unnormalized autocorrelation values. In the other words, we can recover the variance and off-diagonal autocorrelation values, separately.

2. Problem Formulation

Let's assume $\boldsymbol{\tau} \neq \mathbf{0}$, and $\mathbf{x}$ and $\boldsymbol{\tau}$ ($\boldsymbol{\tau} \sim \mathcal{N}(\mathbf{d}, \boldsymbol{\Sigma})$) are independent random vectors. We define a new random process $\mathbf{w}$ such that $\mathbf{w} = \mathbf{x} - \boldsymbol{\tau}$. Clearly, $\mathbf{w}$ is a Gaussian stochastic process $\mathbf{w} \sim \mathcal{N}(-\mathbf{d}, \boldsymbol{P})$ where $\boldsymbol{P} = \boldsymbol{R}_{\mathbf{x}} + \boldsymbol{\Sigma}$. Therefore, with these defaults, we can obtain the main formalism of output autocorrelation function as:

$$R_{\mathbf{y}}(i,j) = R_{\mathbf{y}}(l) = \frac{e^{\frac{-d^2}{p_0+p_l}}}{\pi\sqrt{(p_0^2-p_l^2)}}\left\{\int_0^{\frac{\pi}{2}}\frac{1}{\beta} + \sqrt{\frac{\pi}{\beta}}\frac{\alpha}{2\beta}e^{\frac{\alpha^2}{4\beta}} - \sqrt{\frac{\pi}{\beta}}\frac{\alpha}{\beta}Q\left(\frac{\alpha}{\sqrt{2\beta}}\right)e^{\frac{\alpha^2}{4\beta}}d\theta\right\} - 1. \quad (2)$$

where:

$$\alpha = \frac{d(\sin\theta + \cos\theta)}{p_0+p_l}, \quad \beta = \frac{p_0 - p_l\sin 2\theta}{2(p_0^2-p_l^2)}. \quad (3)$$

Also, p_l and p_0 denote the autocorrelation term for lag l and variance of $\mathbf{w}$, respectively. It remains to evaluate the integral in (2) in terms of p_0 and $\{p_l\}$ which have to be estimated-a task that is central to our efforts in the rest of this paper. Finding p_0 and $\{p_l\}$ results in input variance and autocorrelation recovery, which can be achieved by considering the relation:

$$\boldsymbol{R}_{\mathbf{x}} = \boldsymbol{P} + \boldsymbol{\Sigma}. \quad (4)$$

3. Covariance Recovery Method

Since evaluating the integral in (2) appears to be difficult, we resort to rational approximations to facilitate its evaluation. The Q function is well approximated with the sum of exponentials,

$$Q(x) \approx \frac{1}{12}e^{\frac{-x^2}{2}} + \frac{1}{4}e^{\frac{-2x^2}{3}}, \qquad x > 0. \quad (5)$$

We further note that the integral in (2) may be evaluated by substituting $D_1(\theta; p_0, p_l, d) = \sqrt{\frac{\pi}{\beta}}\frac{\alpha}{\beta}Q\left(\frac{\alpha}{\sqrt{2\beta}}\right)e^{\frac{\alpha^2}{4\beta}}$ and $D_2(\theta; p_0, p_l, d) = \sqrt{\frac{\pi}{\beta}}\frac{\alpha}{2\beta}e^{\frac{\alpha^2}{4\beta}}$ with piece-wise Padé approximants, that yield the best approximation of a function by a rational function of given order through the *moment matching* technique. Thus, D_2 is represented as

$$\theta \in \left[0, \frac{\pi}{8}\right] \cup \left[\frac{3\pi}{8}, \frac{\pi}{2}\right] : \sqrt{\frac{\pi}{\beta}}\frac{\alpha}{2\beta}e^{\frac{\alpha^2}{4\beta}} \approx \frac{e + s\theta}{k + g\theta + h\theta^2},$$
$$\theta \in \left[\frac{\pi}{8}, \frac{3\pi}{8}\right] : \sqrt{\frac{\pi}{\beta}}\frac{\alpha}{2\beta}e^{\frac{\alpha^2}{4\beta}} \approx \frac{z + u\theta + v\theta^2}{k' + g'\theta + h'\theta^2}. \quad (6)$$

Similar approximations can be obtained for terms associated with the function D_1.

p_0 and $\{p_l\}$ are estimated by formulating a minimization problem. For this purpose, one may consider the following criterion:

$$\bar{C}(p_0,p_l) \triangleq \log\left(\left|R_{\mathbf{y}}(l) - \frac{e^{\frac{-d^2}{p_0+p_l}}}{\pi\sqrt{(p_0^2-p_l^2)}}\left\{\int_0^{\frac{\pi}{2}}\frac{1}{\beta} + \sqrt{\frac{\pi}{\beta}}\frac{\alpha}{2\beta}e^{\frac{\alpha^2}{4\beta}} - \sqrt{\frac{\pi}{\beta}}\frac{\alpha}{\beta}Q\left(\frac{\alpha}{\sqrt{2\beta}}\right)e^{\frac{\alpha^2}{4\beta}}d\theta\right\} + 1\right|^2\right) \quad (7)$$

where the autocorrelation of output signal can be estimated with the give sign vector ($\mathbf{y}$) via the sample covariance matrix

$$\boldsymbol{R}_{\mathbf{y}} \approx \frac{1}{N}\sum_{k=1}^{N}\mathbf{y}(k)\mathbf{y}(k)^H \quad (8)$$

Note that by now we have derived an approximated version of (2). Let $H(p_0, p_l)$ denote this approximation. Therefore, we can alternatively consider the criterion:

$$C(p_0,p_l) \triangleq \log\left(|R_{\mathbf{y}}(l) - H(p_0,p_l)|^2\right) \quad (9)$$

To filter out the undesired local minima, we resort to constraints reinforcing the behavior of an autocorrelation function. More precisely, we will consider the minimization problem:

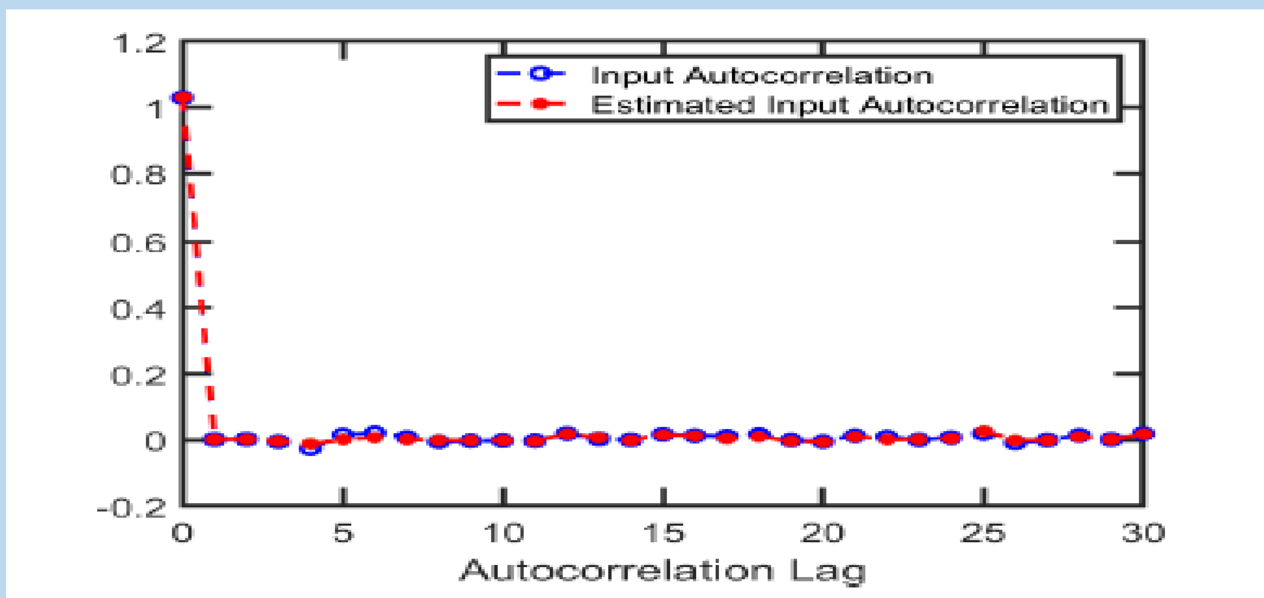
$$\mathcal{P}_\ell : \min_{p_0,p_l} \; C(p_0,p_l), \quad \text{s.t.} \quad p_0^2 \geq p_l^2, \quad p_0 \geq 0 \quad (10)$$

where the first inequality constraint in (10) is imposed to ensure that the magnitude of the diagonal elements of the covariance matrix of $\mathbf{w}$ is greater than the magnitude of the off-diagonal elements. The non-convex problem in (10) may be solved via the gradient descent numerical optimization approach by employing multiple random initial points. Once p_0 and $\{p_l\}$ are obtained, one can estimate the autocorrelation values of $\mathbf{x}$ via (4).

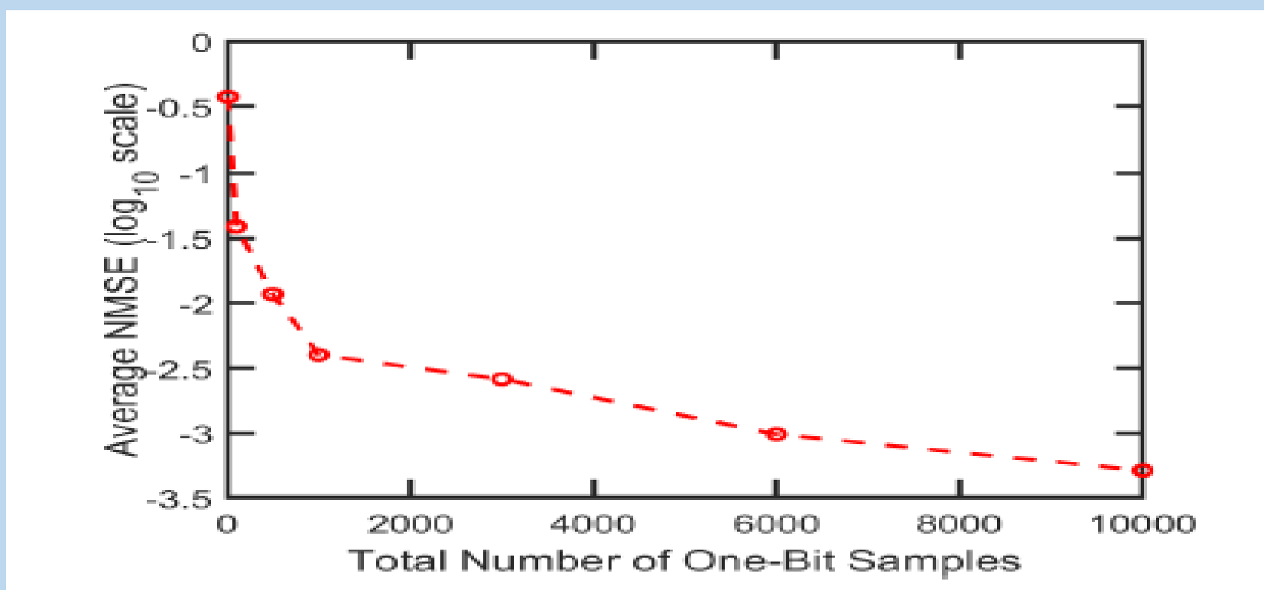
4. Numerical Results

Our results reveal that it can estimate the variance and autocorrelation values, precisely ($d = 0.7, \boldsymbol{\Sigma} = 0.3\boldsymbol{I}$).

Autocorrelation sequence recovery for random sequence of length 31:



The impact of a growing sample size in the variance recovery:



5. Conclusion

We proposed a modified arcsine law through Padé approximations that can make use of non-zero time-varying thresholds in one-bit sampling. Also, the numerical results showcase the effectiveness of the proposed approach in recovering the autocorrelation values of one-bit sampled stationary signals.