TECHNISCHE UNIVERSITÄT DARMSTADT

T3 – CHARACTERIZING AMBIGUITIES IN SPARSE ARRAYS

IEEE SAM Workshop

Marius Pesavento

TU Darmstadt

July 13, 2026

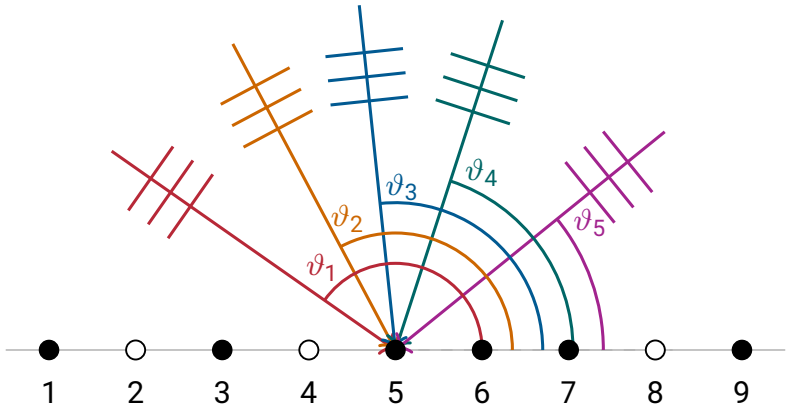
pesavento@ieee.org

Open6GHub+

TECHNISCHE UNIVERSITÄT DARMSTADT

SPARSE ARRAYS

Thinned arrays



TECHNISCHE UNIVERSITÄT DARMSTADT

SPARSE ARRAYS

Sparse arrays offer several benefits:

- Large aperture
- Low hardware cost/ complexity

How to optimally design sparse arrays?

- Optimized beampattern (Mainlobe width vs sidelobe/grating level)
- Optimized performance bounds (CRB,...)[1][2]
- Optimized co-array properties [3][4][5]

Problem with these designs:

- Beampattern and Performance bounds do not capture the essential aspects of the complex-valued interplay between multiple-sources.
- Performance bounds are often local.

[1] J. Tabrikian et al., "Cognitive Antenna Selection for Automotive Radar Using Bobrovsky-Zakai Bound", 2021

[2] I. van der Werf et al., "Jointly Optimal Array Geometries and Waveforms in Active Sensing: New Insights Into Array Design via the Cramér-Rao Bound", 2025

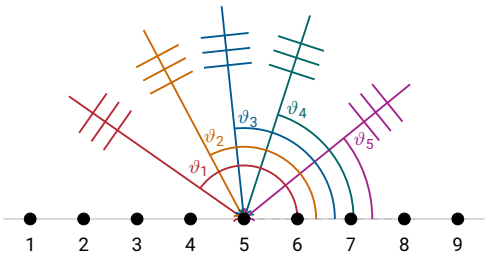
[3] P. Pal and P. P. Vaidyanathan, "Nested Arrays: A Novel Approach to Array Processing With Enhanced Degrees of Freedom", 2010

[4] P. P. Vaidyanathan and P. Pal, "Sparse Sensing With Co-Prime Samplers and Arrays", 2011

[5] X. Wang et al., "Analysis and Design of Optimum Sparse Array Configurations for Adaptive Beamforming", 2018

TECHNISCHE UNIVERSITÄT DARMSTADT

DOA ESTIMATION IN UNIFORM LINEAR ARRAYS



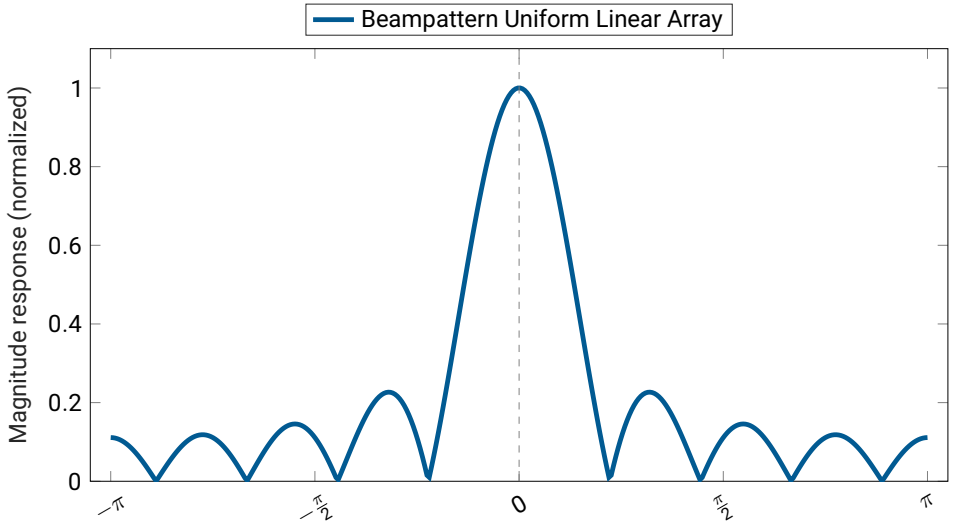
$$\boldsymbol{A}(z_1, z_2, \cdots, z_5) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ z_1 & z_2 & z_3 & z_4 & z_5 \\ z_1^2 & z_2^2 & z_3^2 & z_4^2 & z_5^2 \\ z_1^3 & z_2^3 & z_3^3 & z_4^3 & z_5^3 \\ z_1^4 & z_2^4 & z_3^4 & z_4^4 & z_5^4 \\ z_1^5 & z_2^5 & z_3^5 & z_4^5 & z_5^5 \\ z_1^6 & z_2^6 & z_3^6 & z_4^6 & z_5^6 \\ z_1^7 & z_2^7 & z_3^7 & z_4^7 & z_5^7 \\ z_1^8 & z_2^8 & z_3^8 & z_4^8 & z_5^8 \end{bmatrix}$$

$$\boldsymbol{y} = \boldsymbol{A}(z_1, \ldots, z_L)\,\boldsymbol{x} \quad + \text{noise}$$

- $\boldsymbol{y} = [y_1, \ldots y_M]^{\mathsf{T}}$ are the sensor measurements
- $\boldsymbol{x} = [x_1, \ldots x_L]^{\mathsf{T}}$ is the signal waveform
- $\boldsymbol{A}(z_1, \ldots, z_L) = [\boldsymbol{a}(z_1), \ldots, \boldsymbol{a}(z_L)]$ is a full $M \times L$ Vandermonde matrix
- Generators $z_\ell = e^{\mathrm{j}2\pi\theta_\ell}$
- $\theta_\ell = \frac{2\pi d}{\lambda} \cos\vartheta_\ell$ are the electrical angles corresponding to ϑ_ℓ.

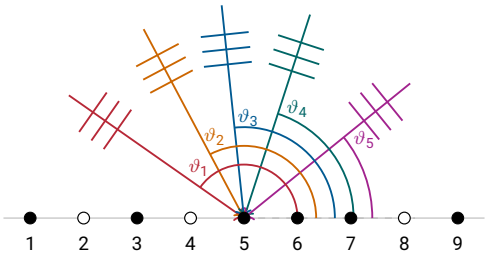
TECHNISCHE UNIVERSITÄT DARMSTADT

GRATING LOBES IN SPARSE ARRAYS



TECHNISCHE UNIVERSITÄT DARMSTADT

DOA ESTIMATION IN THINNED UNIFORM LINEAR ARRAYS



$$\boldsymbol{A}_{\mathcal{I}}(z_1, z_2, \cdots, z_5) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ z_1 & z_2 & z_3 & z_4 & z_5 \\ z_1^2 & z_2^2 & z_3^2 & z_4^2 & z_5^2 \\ z_1^3 & z_2^3 & z_3^3 & z_4^3 & z_5^3 \\ z_1^4 & z_2^4 & z_3^4 & z_4^4 & z_5^4 \\ z_1^5 & z_2^5 & z_3^5 & z_4^5 & z_5^5 \\ z_1^6 & z_2^6 & z_3^6 & z_4^6 & z_5^6 \\ z_1^7 & z_2^7 & z_3^7 & z_4^7 & z_5^7 \\ z_1^8 & z_2^8 & z_3^8 & z_4^8 & z_5^8 \end{bmatrix}$$

$$\boldsymbol{y}_{\mathcal{I}} = \boldsymbol{A}_{\mathcal{I}}(z_1, \ldots, z_L)\,\boldsymbol{x} \quad + \text{noise}$$

- Index set of sensors kept:

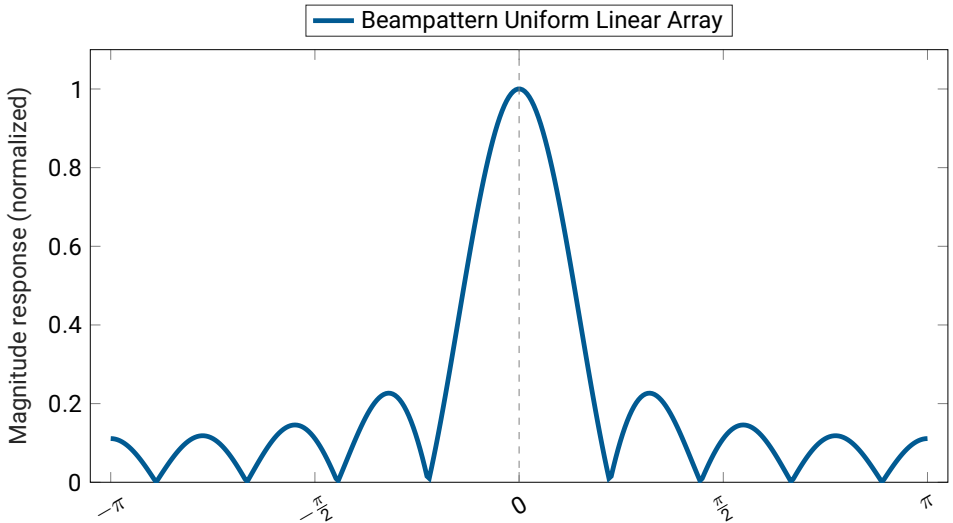
$$\mathcal{I} = \{1, 3, 5, 6, 7, 9\},$$

- Complementary index set of removed sensors: complementary set

$$\bar{\mathcal{I}} = \{2, 4, 8\},$$

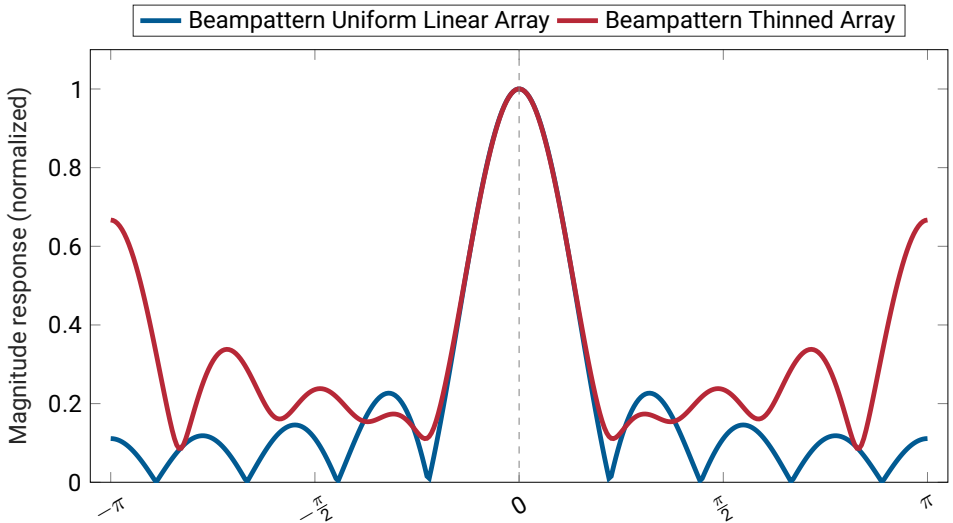
TECHNISCHE UNIVERSITÄT DARMSTADT

GRATING LOBES IN SPARSE ARRAYS



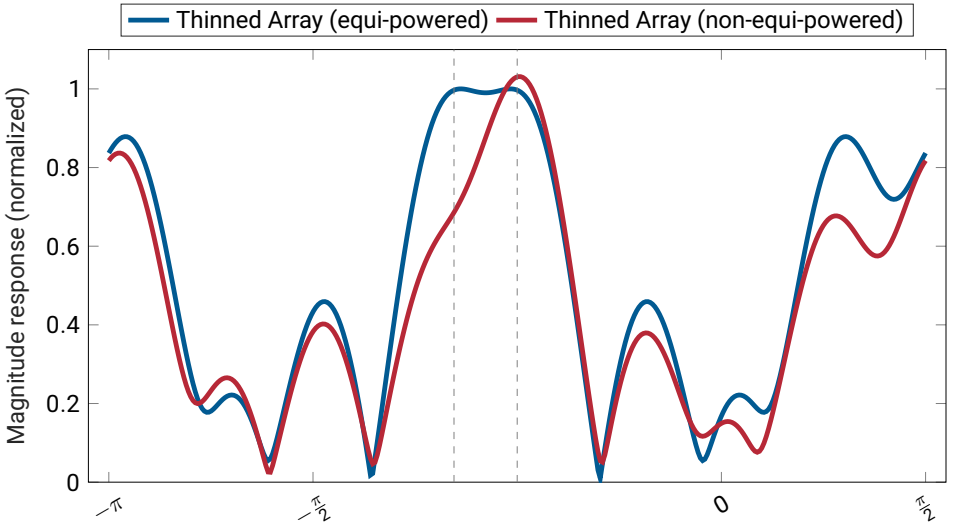
TECHNISCHE UNIVERSITÄT DARMSTADT

GRATING LOBES IN SPARSE ARRAYS



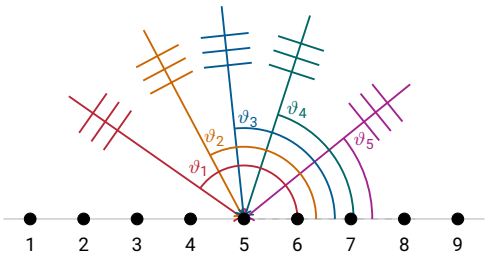
TECHNISCHE UNIVERSITÄT DARMSTADT

GRATING LOBES IN SPARSE ARRAYS



TECHNISCHE UNIVERSITÄT DARMSTADT

RANK PROPERTY OF STEERING MATRIX AND UNIQUENESS



$$\boldsymbol{A}(z_1, z_2, \cdots, z_5) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ z_1 & z_2 & z_3 & z_4 & z_5 \\ z_1^2 & z_2^2 & z_3^2 & z_4^2 & z_5^2 \\ z_1^3 & z_2^3 & z_3^3 & z_4^3 & z_5^3 \\ z_1^4 & z_2^4 & z_3^4 & z_4^4 & z_5^4 \\ z_1^5 & z_2^5 & z_3^5 & z_4^5 & z_5^5 \\ z_1^6 & z_2^6 & z_3^6 & z_4^6 & z_5^6 \\ z_1^7 & z_2^7 & z_3^7 & z_4^7 & z_5^7 \\ z_1^8 & z_2^8 & z_3^8 & z_4^8 & z_5^8 \end{bmatrix}$$

$$\boldsymbol{y} = \boldsymbol{A}(z_1, \ldots, z_L)\,\boldsymbol{x} \quad + \text{noise}$$

Full Vandermonde matrix $\boldsymbol{A}(z_1, z_2, z_3 \cdots, z_L)$ is always full rank.

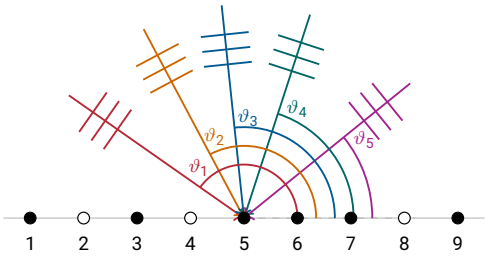
$\Downarrow$

Parameter estimates are unique.[6]

[6] M. Wax and I. Ziskind, "On unique localization of multiple sources by passive sensor arrays", Jul. 1989

TECHNISCHE UNIVERSITÄT DARMSTADT

RANK PROPERTY OF STEERING MATRIX AND UNIQUENESS



$$\boldsymbol{A}_{\mathcal{I}}(z_1, z_2, \cdots, z_5) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ z_1 & z_2 & z_3 & z_4 & z_5 \\ z_1^2 & z_2^2 & z_3^2 & z_4^2 & z_5^2 \\ z_1^3 & z_2^3 & z_3^3 & z_4^3 & z_5^3 \\ z_1^4 & z_2^4 & z_3^4 & z_4^4 & z_5^4 \\ z_1^5 & z_2^5 & z_3^5 & z_4^5 & z_5^5 \\ z_1^6 & z_2^6 & z_3^6 & z_4^6 & z_5^6 \\ z_1^7 & z_2^7 & z_3^7 & z_4^7 & z_5^7 \\ z_1^8 & z_2^8 & z_3^8 & z_4^8 & z_5^8 \end{bmatrix}$$

- Reversely, if the thinned steering matrix $\boldsymbol{A}_{\mathcal{I}}(z_1, z_2, z_3 \cdots, z_5)$ drops rank $\Leftrightarrow$ there exist two different parameter sets $\{z_1, z_2, \boldsymbol{x}_1\}$ and $\{z_3, z_4, z_5, \boldsymbol{x}_2\}$ such that:[7][8][9]

$$\boldsymbol{y}_{\mathcal{I}} = \boldsymbol{A}_{\mathcal{I}}(z_1, z_2)\, \boldsymbol{x}_1 = -\boldsymbol{A}_{\mathcal{I}}(z_3, z_4, z_5)\, \boldsymbol{x}_2$$

[7] A. Manikas and C. Proukakis, "Modeling and estimation of ambiguities in linear arrays", Aug. 1998

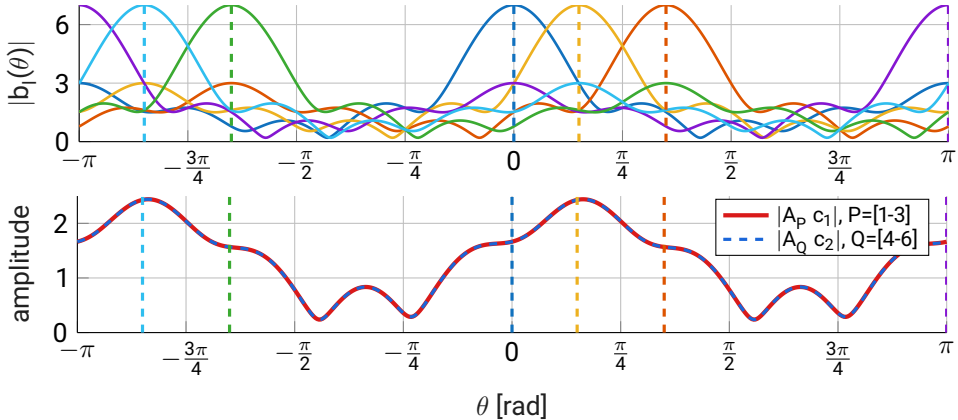
[8] Y. Abramovich et al., "Identifiability and manifold ambiguity in DOA estimation for nonuniform linear antenna arrays", Mar. 1999

[9] B. Friedlander, "Multi-Signal Ambiguity in Direction-of-Arrival Estimation", 2023

TECHNISCHE UNIVERSITÄT DARMSTADT

BEAMPATTERN FOR AMBIGUOUS ARRAY

Example 6 of http://arxiv.org/abs/2606.00360



TECHNISCHE UNIVERSITÄT DARMSTADT

TABLE OF CONTENTS

Introduction
- Rank conditions and identifiability
- Array ambiguities

Sparse array analysis
- State of the art
- Generalized Vandermonde rank conditions
- Alternative rank criterion with fixed generators

Sparse array design
- Coarrays
- Ambiguity-based array design

Conclusions and outlook

Introduction

Rank conditions and identifiability

SINGLE SNAPSHOT / COHERENT SOURCE CASE

How is the number of uniquely identifiable sources related to the Kruskal rank of the steering matrix?

- Let K_{rank} denote the Kruskal rank of the steering matrix $\boldsymbol{A}_{\mathcal{I}}(z_1, \ldots, z_{K_{\text{rank}}})$, i.e., the largest number of columns with distinct generators, for which the steering matrix is always full column rank.
- Nullspace property: There exists $K_{\text{rank}} + 1$ distinct generators $z_1, \ldots, z_{K_{\text{rank}}}, z_{K_{\text{rank}}+1}$ and linear coefficients $x_1, \ldots, x_{K_{\text{rank}}}, x_{K_{\text{rank}}+1}$ such that

$$[\boldsymbol{A}(z_1, \ldots, z_{K_{\text{rank}}}) \;,\; \boldsymbol{a}(z_{K_{\text{rank}}+1})] \begin{bmatrix} x_1 \\ \vdots \\ x_{K_{\text{rank}}} \\ x_{K_{\text{rank}}+1} \end{bmatrix} = \boldsymbol{0}$$

- Spark: The $\text{spark}\,(\boldsymbol{A}) = K_{\text{rank}} + 1$ is defined as the smallest number of columns of an arbitrary steering matrix $\boldsymbol{A}$ with distinct generators, that are linearly dependent.

SINGLE SNAPSHOT / COHERENT SOURCE CASE

How is the number of uniquely identifiable sources related to the Kruskal rank of the steering matrix?

- Hence for $L' = \lceil \frac{K_{\text{rank}}+1}{2} \rceil$ the set of generators can be partitioned as $z_1, \ldots, z_{L'}, z_{L'+1}, \ldots, z_{K_{\text{rank}}+1}$ such that

$$\boldsymbol{y} = \boldsymbol{A}(z_1, \ldots, z_{L'}) \begin{bmatrix} x_1 \\ \vdots \\ x_{L'} \end{bmatrix} = -\boldsymbol{A}(z_{L'+1}, \ldots, \ldots, z_{K_{\text{rank}}+1}) \begin{bmatrix} x_{L'} \\ \vdots \\ x_{K_{\text{rank}}+1} \end{bmatrix}$$

- In other words, the measurements $\boldsymbol{y}$ can indistinguishably be explained by the two distinct parameter sets

$$\{z_1, \ldots, z_{L'}, x_1, \ldots, x_{L'}\} \neq \{z_{L'}, \ldots, z_{K_{\text{rank}}+1}, x_{L'}, \ldots, x_{K_{\text{rank}}+1}\}$$

- Thus, in the single snapshot / coherent source case, the number of uniquely identifiable sources[6]

$$L' < \left\lceil \frac{K_{\text{rank}}+1}{2} \right\rceil .$$

[6] M. Wax and I. Ziskind, "On unique localization of multiple sources by passive sensor arrays", Jul. 1989

η-NON-COHERENT WAVEFORM VECTORS / SOURCES

How is the number of uniquely identifiable sources related to the Kruskal rank of the steering matrix?

- With K_{rank} denoting the Kruskal rank of the steering matrix $\boldsymbol{A}_{\mathcal{I}}(z_1, \ldots, z_{K_{\text{rank}}})$.
- Nullspace property: There exist generators $z_1, \ldots, z_{K_{\text{rank}}}, z_{K_{\text{rank}}+1}, \ldots, z_{K_{\text{rank}}+\eta}$ and linear coefficients $x_1(t_i), \ldots, x_{K_{\text{rank}}}(t_i), x_{K_{\text{rank}}+1}(t_i), \ldots, x_{K_{\text{rank}}+\eta}(t_i)$ for $\eta \leq K_{\text{rank}}$ and $t_i \in \{t_1, \ldots, t_\eta\}$ such that

$$\underbrace{\left[\underbrace{\boldsymbol{A}(z_1, \ldots, z_{K_{\text{rank}}})}_{\text{linearly independent}}, \; \boldsymbol{a}(z_{K_{\text{rank}}+1}), \ldots, \boldsymbol{a}(z_{K_{\text{rank}}+\eta})\right]}_{\text{Nullspace dimension}=\eta} \underbrace{\begin{bmatrix} x_1(t_1) & \cdots & x_1(t_\eta) \\ \vdots & \vdots & \vdots \\ x_{K_{\text{rank}}}(t_1) & \cdots & x_{K_{\text{rank}}}(t_\eta) \\ x_{K_{\text{rank}}+1}(t_1) & \cdots & x_{K_{\text{rank}}+1}(t_\eta) \\ \vdots & \vdots & \vdots \\ x_{K_{\text{rank}}+\eta}(t_1) & \cdots & x_{K_{\text{rank}}+\eta}(t_\eta) \end{bmatrix}}_{\eta \text{ linearly independent waveform vectors}} = \boldsymbol{0}$$

η-NON-COHERENT WAVEFORM VECTORS / SOURCES

How is the number of uniquely identifiable sources related to the Kruskal rank of the steering matrix?

- As before, now for $L' = \lceil \frac{K_{\text{rank}}+\eta}{2} \rceil$ with $\eta \leq K_{\text{rank}}$ the set of generators can be partitioned as $z_1, \ldots, z_{L'}, z_{L'+1}, \ldots, z_{K_{\text{rank}}+\eta}$ such that

$$\boldsymbol{y} = \boldsymbol{A}(z_1, \ldots, z_{L'}) \begin{bmatrix} x_1(t_1) & \cdots & x_1(t_\eta) \\ \vdots & \vdots & \vdots \\ x_{L'}(t_1) & \cdots & x_{L'}(t_\eta) \end{bmatrix} = -\boldsymbol{A}(z_{L'+1}, \ldots, z_{K_{\text{rank}}+\eta}) \begin{bmatrix} x_{L'}(t_1) & \cdots & x_{L'}(t_\eta) \\ \vdots & \vdots & \vdots \\ x_{K_{\text{rank}}+\eta}(t_1) & \cdots & x_{K_{\text{rank}}+\eta}(t_\eta) \end{bmatrix}$$

- Hence, the measurements $\boldsymbol{y}$ are indistinguishable under the two distinct parameter sets

$$\{z_1, \ldots, z_{L'}, x_1(t_1), \ldots, x_{L'}(t_\eta)\} \neq \{z_{L'}, \ldots, z_{K_{\text{rank}}+\eta}, x_{L'}(t_i), \ldots, x_{K_{\text{rank}}+\eta}(t_i)\}$$

- Thus, in the η non-coherent waveform /snapshot / source case, the number of uniquely identifiable sources[6]

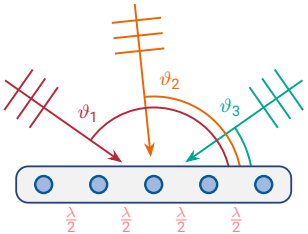
$$L' < \left\lceil \frac{K_{\text{rank}} + \eta}{2} \right\rceil .$$

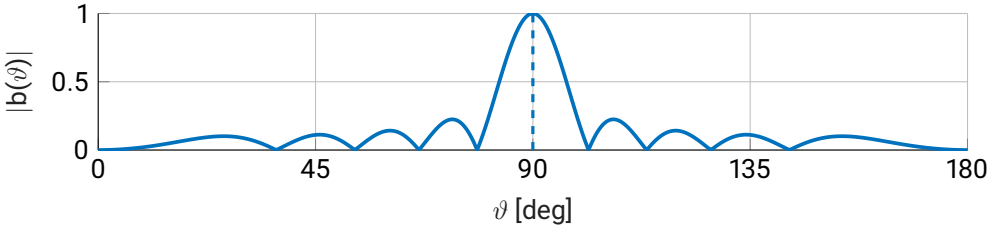
[6] M. Wax and I. Ziskind, "On unique localization of multiple sources by passive sensor arrays", Jul. 1989

Introduction

Array ambiguities

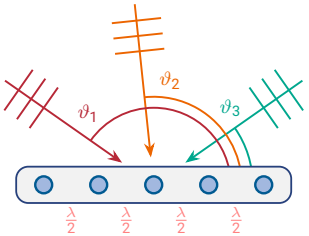
FIRST-ORDER AMBIGUITIES: GRATING LOBES

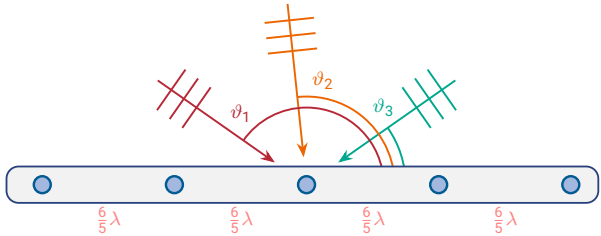


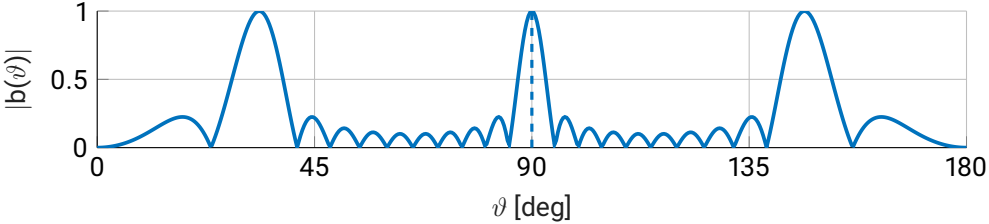


TECHNISCHE UNIVERSITÄT DARMSTADT

FIRST-ORDER AMBIGUITIES: SPATIAL SAMPLING THEOREM

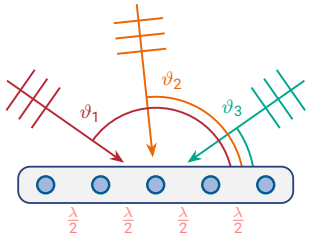


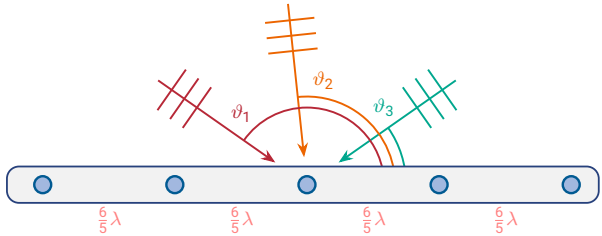




TECHNISCHE UNIVERSITÄT DARMSTADT

FIRST-ORDER AMBIGUITIES: GRATING LOBES





- Array steering vectors for different directions become identical, i.e., for $d = \frac{6}{5}\lambda$

$$\boldsymbol{a}(\vartheta) = \begin{bmatrix} 1 \\ e^{\mathrm{j}\frac{2\pi d}{\lambda}\cos\vartheta_\ell} \\ e^{\mathrm{j}\frac{4\pi d}{\lambda}\cos\vartheta_\ell} \\ \cdots \\ e^{\mathrm{j}\frac{2\pi(M-1)d}{\lambda}\cos\vartheta_\ell} \end{bmatrix} = \begin{bmatrix} 1 \\ e^{\mathrm{j}2\pi\frac{6}{5}\cos\vartheta_\ell} \\ e^{\mathrm{j}2\pi 2\frac{6}{5}\cos\vartheta_\ell} \\ \cdots \\ e^{\mathrm{j}2\pi(M-1)\frac{6}{5}\cos\vartheta_\ell} \end{bmatrix} \Rightarrow \boldsymbol{a}(\vartheta)\,|_{\vartheta=90^\circ} = \boldsymbol{a}(\vartheta)\,|_{\vartheta=\arccos\frac{5}{6}\approx 33.56^\circ \wedge 146.44^\circ} = \mathbf{1}$$

FIRST-ORDER AMBIGUITIES: GRATING LOBES

- For first-order ambiguities with $\boldsymbol{a}(\vartheta') = \boldsymbol{a}(\vartheta'')$ the corresponding array steering matrix

$$\boldsymbol{A}(\vartheta', \vartheta'', \vartheta_3, \ldots, \vartheta_L) = [\boldsymbol{a}(\vartheta'), \boldsymbol{a}(\vartheta''), \boldsymbol{a}(\vartheta_3), \ldots, \boldsymbol{a}(\vartheta_L)]$$

  has two (or multiple) identical columns, hence the matrix is rank deficient.
- First-order ambiguities are trivial and comparably easy to detect and handle, e.g., by limiting the field of view.
- In practice, first-order ambiguities may also be suppressed by the superimposed element pattern.

HIGHER-ORDER AMBIGUITIES

- Higher-order ambiguities emerge in the case of non-trivial rank drop conditions of the array steering matrix

$$\boldsymbol{A}_{\mathcal{I}}(\vartheta_1, \vartheta_2, \ldots, \vartheta_L) = [\boldsymbol{a}_{\mathcal{I}}(\vartheta_1), \boldsymbol{a}_{\mathcal{I}}(\vartheta_2), \ldots, \boldsymbol{a}_{\mathcal{I}}(\vartheta_L)]\,,$$

  i.e., when $L \leq M_{\mathcal{I}}$ distinct steering vectors $\boldsymbol{a}_{\mathcal{I}}(\vartheta_1) \neq \boldsymbol{a}_{\mathcal{I}}(\vartheta_2) \neq \ldots, \neq, \boldsymbol{a}_{\mathcal{I}}(\vartheta_L)$ become linearly dependent.

- In this case there exists a linear combination with coefficients $\alpha_1, \alpha_2, \ldots, \alpha_L$ such that $\sum_{\ell=1}^{L} \alpha_\ell \, \boldsymbol{a}_{\mathcal{I}}(\vartheta_\ell) = \boldsymbol{0}$, hence the vector $\boldsymbol{\alpha} = [\alpha_1, \alpha_2, \ldots, \alpha_L]^{\mathsf{T}}$ lies in the nullspace of $\boldsymbol{A}_{\mathcal{I}}(\vartheta_1, \vartheta_2, \ldots, \vartheta_L)$, i.e., the nullspace

$$\mathcal{N}\big(\boldsymbol{A}_{\mathcal{I}}(\vartheta_1, \vartheta_2, \ldots, \vartheta_L)\big) = \{\boldsymbol{\gamma} \in \mathbb{C}^L \mid \boldsymbol{A}_{\mathcal{I}}(\vartheta_1, \vartheta_2, \ldots, \vartheta_L)\, \boldsymbol{\gamma} = \boldsymbol{0}\}$$

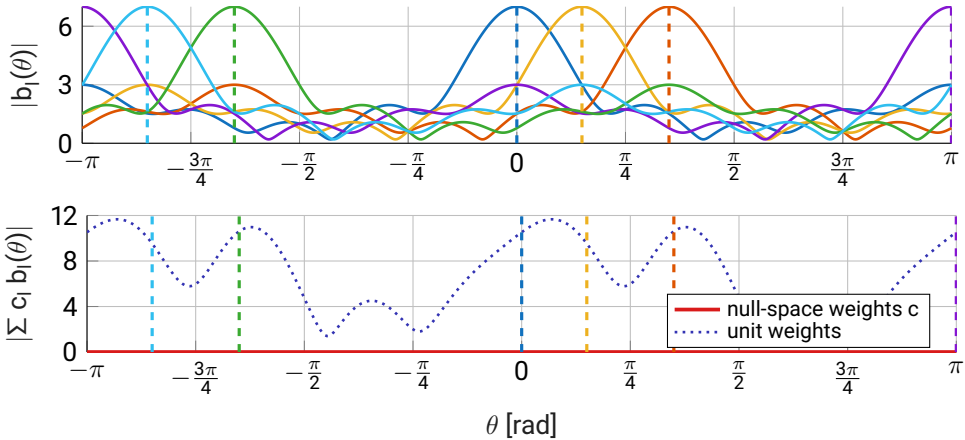
  is non-empty and

$$\boldsymbol{\alpha} \in \mathcal{N}\big(\boldsymbol{A}_{\mathcal{I}}(\vartheta_1, \vartheta_2, \ldots, \vartheta_L)\big).$$

- Determining the set of all higher-order ambiguities for arbitrary arrays or for sparse (thinned) linear arrays is non-trivial. In the following we focus on thinned ULAs.

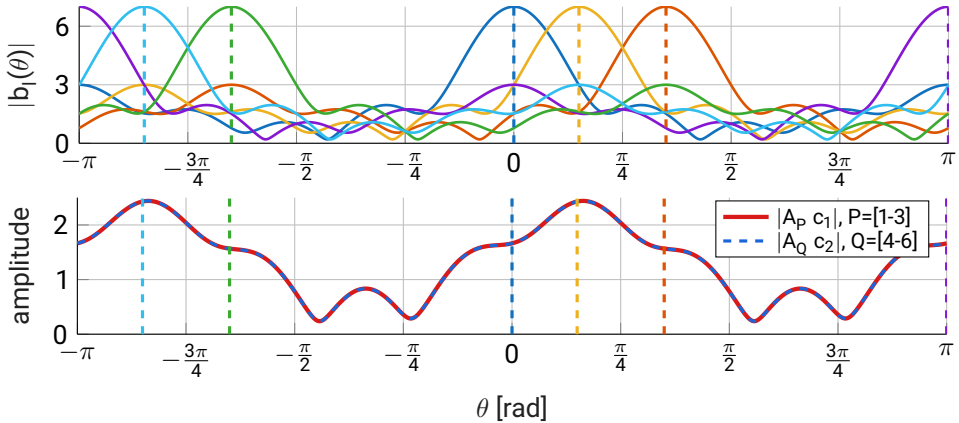
TECHNISCHE UNIVERSITÄT DARMSTADT

BEAMPATTERNS: EXAMPLE 6



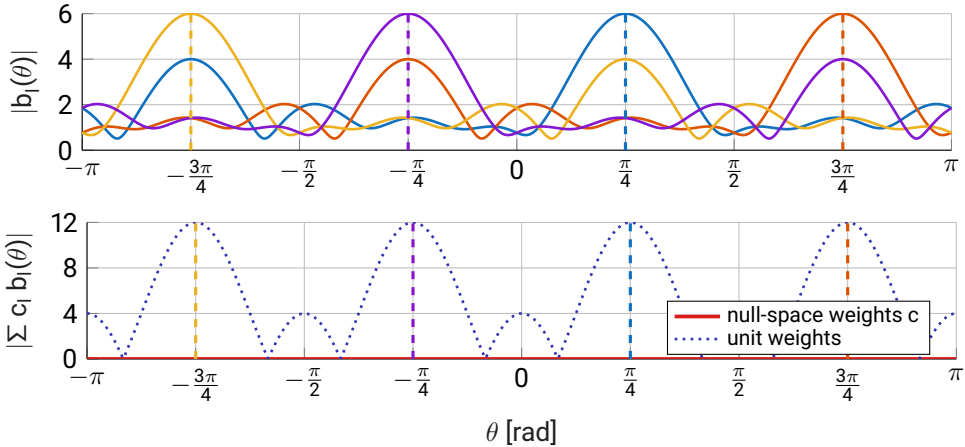
TECHNISCHE UNIVERSITÄT DARMSTADT

BEAMPATTERNS: EXAMPLE 6



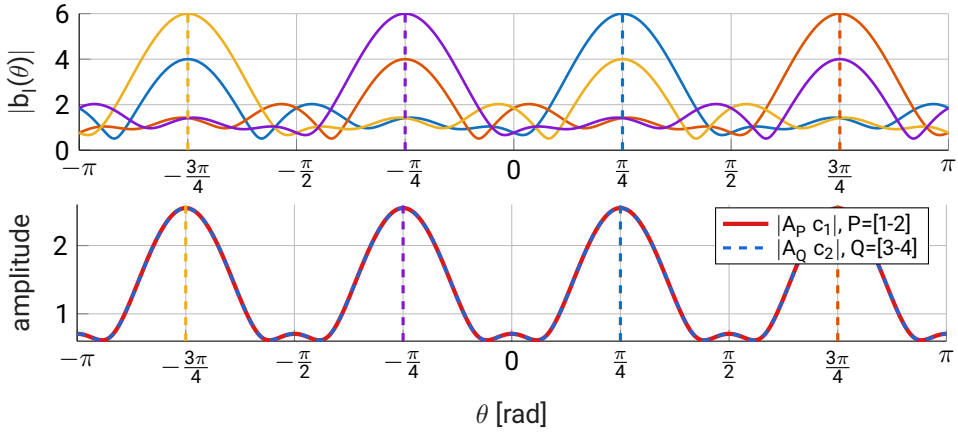
TECHNISCHE UNIVERSITÄT DARMSTADT

BEAMPATTERNS: EXAMPLE 7



TECHNISCHE UNIVERSITÄT DARMSTADT

BEAMPATTERNS: EXAMPLE 7



Sparse array analysis

State of the art

TECHNISCHE UNIVERSITÄT DARMSTADT

SPARSE ARRAYS: THINNED ULA

- Uniform Linear Array (ULA): Sensors on a regular grid with equal inter-sensor spacing
- Sparse arrays: Sensors are removed, leaving irregular gaps in the grid

TECHNISCHE UNIVERSITÄT DARMSTADT

SPARSE ARRAYS: UNIFORM AMBIGUITIES

Existing results

- Manikas and Proukakis identified a set of uniformly (in electrical angle) spaced ambiguity generators, for which the steering matrix drops rank.
- Given the sensor locations corresponding to the index set

$$\mathcal{I} = \{m_1, m_2, \ldots, m_{M_{\mathcal{I}}}\}, \quad 1 = m_1 < m_2 < \cdots < m_{M_{\mathcal{I}}} = M$$

ambiguities emerge if a generator set

$$\{1, e^{\mathrm{j}\pi \frac{2}{|m_i - m_j|}}, e^{\mathrm{j}\pi \frac{4}{|m_i - m_j|}}, \ldots, e^{\mathrm{j}\pi \frac{2c}{|m_i - m_j|}}\}$$

exists, where $i \neq j \in [1, \ldots, M]$ and $c \in \mathbb{N}$ is the largest integer satisfying $c < |m_i - m_j|$ and if the set contains at least M elements, i.e., $c \geq (M-1)$. [7]

[7] A. Manikas and C. Proukakis, "Modeling and estimation of ambiguities in linear arrays", Aug. 1998

SPARSE ARRAYS: UNIFORM AMBIGUITIES

Existing results

- To see that M generators from the set

$$\{1, e^{\mathrm{j}\pi \frac{2}{|m_i - m_j|}}, e^{\mathrm{j}\pi \frac{4}{|m_i - m_j|}}, \ldots, e^{\mathrm{j}\pi \frac{2c}{|m_i - m_j|}}\}$$

  lead to a rank drop in the array steering matrix $\boldsymbol{A}$, we choose without loss of generality m_i as the origin, i.e., $m_i = 0$.

- Then the m_i-th and the m_j-th row of the steering matrix are given by

$$[1, e^{\mathrm{j}0\pi \frac{2}{|m_j|}}, e^{\mathrm{j}0\pi \frac{4}{|m_j|}}, \ldots, e^{\mathrm{j}0\pi \frac{2c}{|m_j|}}]^{\mathsf{T}} = [1, 1, 1, \ldots, 1]^{\mathsf{T}}$$

  and

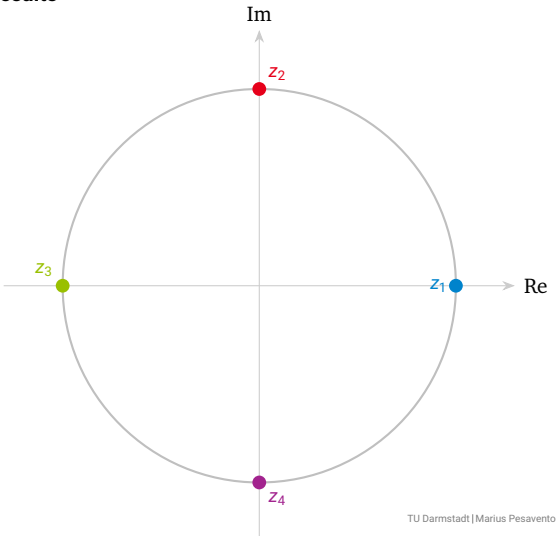
$$[1, e^{\mathrm{j}m_j\pi \frac{2}{|m_j|}}, e^{\mathrm{j}m_j\pi \frac{4}{|m_j|}}, \ldots, e^{\mathrm{j}m_j\pi \frac{2c}{|m_j|}}]^{\mathsf{T}} = [1, 1, 1, \ldots, 1]^{\mathsf{T}}$$

  respectively. Hence, the rows become identical and the steering matrix drops rank.

TECHNISCHE UNIVERSITÄT DARMSTADT

UNIFORM AMBIGUITIES

Existing results



Uniform generator set:

$$\arg z_1 = 0, \quad \arg z_2 = \tfrac{\pi}{2},$$

$$\arg z_3 = \pi, \quad \arg z_4 = \tfrac{3\pi}{2},$$

Any 3 columns of matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & e^{\mathrm{j}\frac{\pi}{2}} & e^{\mathrm{j}\pi} & e^{\mathrm{j}\frac{3\pi}{2}} \\ 1 & e^{\mathrm{j}4\frac{\pi}{2}} & e^{\mathrm{j}4\pi} & e^{\mathrm{j}4\frac{3\pi}{2}} \end{bmatrix}$$

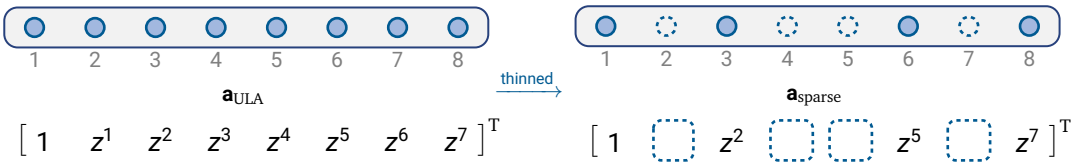
are linearly dependent.

TECHNISCHE UNIVERSITÄT DARMSTADT

SPARSE SYMMETRIC ARRAY: THINNED ULA

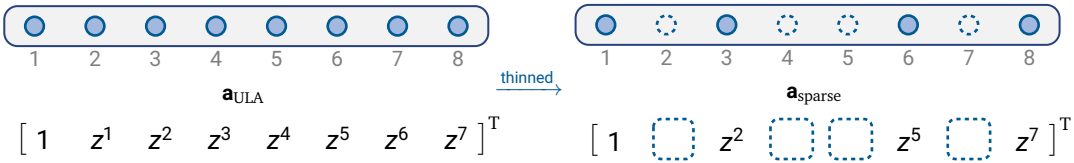
Existing results

- Uniform Linear Array (ULA): Sensors on a regular grid with equal inter-sensor spacing
- Sparse arrays: Sensors are removed at symmetric positions around the center, leaving irregular gaps in the grid



NON-UNIFORM AMBIGUITIES

Existing results



- If the (square) sparse array steering matrix

$$\boldsymbol{A}_{\mathcal{I}}(z_1,\ldots,z_4) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ z_1^2 & z_2^2 & z_3^2 & z_4^2 \\ z_1^5 & z_2^5 & z_3^5 & z_4^5 \\ z_1^7 & z_2^7 & z_3^7 & z_4^7 \end{bmatrix}$$

drops rank, then there must exist a non-zero nullspace vector $\boldsymbol{p}$ such that $\boldsymbol{A}_{\mathcal{I}}^{\mathsf{T}}\boldsymbol{p} = \boldsymbol{0}$, i.e., $\boldsymbol{p} \in \mathcal{N}\big(\boldsymbol{A}_{\mathcal{I}}^{\mathsf{T}}(z_1,\ldots,z_4)\big)$.

NON-UNIFORM AMBIGUITIES

Existing results

- Manikas and Proukakis[7] and later Dowlut[10] reversed this argument. They constructed particular vectors $\{\boldsymbol{p}_1, \ldots, \boldsymbol{p}_{M_\mathcal{I}}\}$, motivated by differential geometry, and tested if corresponding structure nullspace matrices $\boldsymbol{A}_\mathcal{I}(z_{r,1}, \ldots, z_{r,L})$ exist, for which

$$\boldsymbol{A}_\mathcal{I}^\mathsf{T}(z_{r,1}, \ldots, z_{r,L})\boldsymbol{p}_r = \boldsymbol{0}, \quad r = 1, \ldots, M_\mathcal{I}$$

- This conditions essentially requires that the symmetric sparse steering vectors $\boldsymbol{a}_\mathcal{I}(z)$ for all $z \in \{z_{r,1}, \ldots, z_{r,L}\}$ satisfy

$$P_r(z) = \boldsymbol{a}_\mathcal{I}^\mathsf{T}(z)\boldsymbol{p}_r = \sum_{\ell=0}^{L-1} p_{r,\ell+1} z^\ell = 0.$$

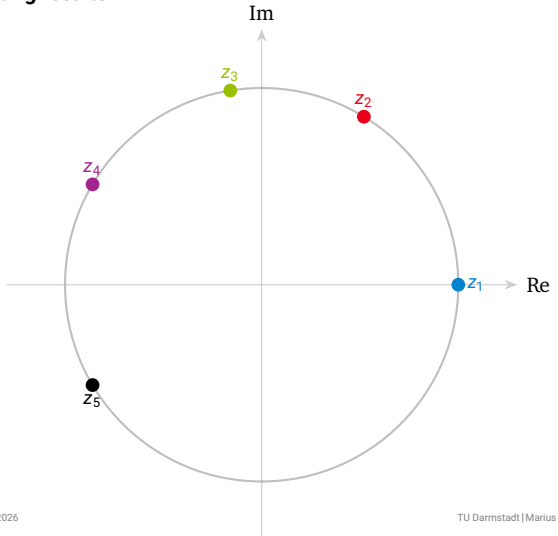
- Hence, the roots of $P_r(z)$, denoted by $\{z_{r,1}, \ldots, z_{r,L}\}$ form a generator set for which the corresponding steering matrix $\boldsymbol{A}_\mathcal{I}(z_{r,1}, \ldots, z_{r,L})$ becomes rank deficient.

[7] A. Manikas and C. Proukakis, "Modeling and estimation of ambiguities in linear arrays", Aug. 1998

[10] N. Dowlut, "An extended ambiguity criterion for array design", 2002

NON-UNIFORM AMBIGUITY SET 1

Existing results



Inter-element spacing: $0.6\,\lambda/2$

Ambiguous spatial frequencies:

$$\arg z_1 = 0, \quad \arg z_2 = 1.0241,$$

$$\arg z_3 = 1.7312, \quad \arg z_4 = 2.6064,$$

$$\arg z_5 = -2.6064,$$

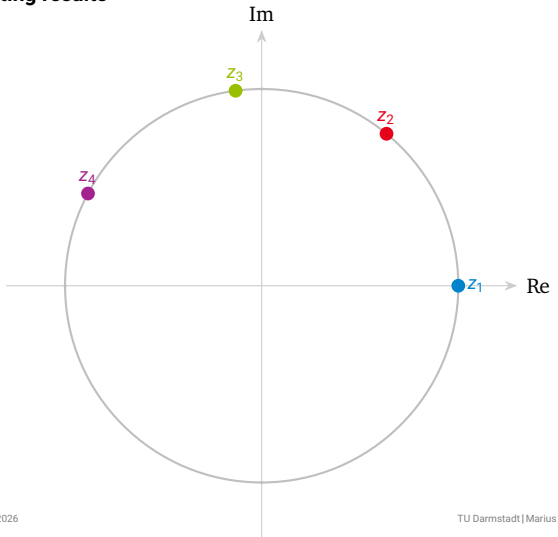
Any 4 columns of matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & z_2^2 & z_3^2 & z_4^2 & z_5^2 \\ 1 & z_2^5 & z_3^5 & z_4^5 & z_5^5 \\ 1 & z_2^7 & z_3^7 & z_4^7 & z_5^7 \end{bmatrix}$$

are linearly dependent.

NON-UNIFORM AMBIGUITY SET 2

Existing results



Inter-element spacing: $0.6\,\lambda/2$

Ambiguous spatial frequencies:

$$\arg z_1 = 0, \quad \arg z_2 = 0.8830,$$

$$\arg z_3 = 1.7038, \quad \arg z_4 = 2.6536,$$

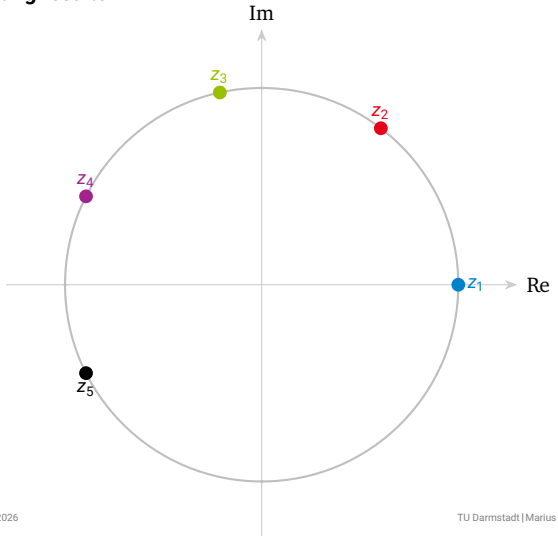
The columns of matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & z_2^2 & z_3^2 & z_4^2 \\ 1 & z_2^5 & z_3^5 & z_4^5 \\ 1 & z_2^7 & z_3^7 & z_4^7 \end{bmatrix}$$

are linearly dependent.

NON-UNIFORM AMBIGUITY SET 3

Existing results



Inter-element spacing: $0.6\,\lambda/2$

Ambiguous spatial frequencies:

$$\arg z_1 = 0, \quad \arg z_2 = 0.9197,$$

$$\arg z_3 = 1.7851, \quad \arg z_4 = 2.6756,$$

$$\arg z_5 = -2.6756,$$

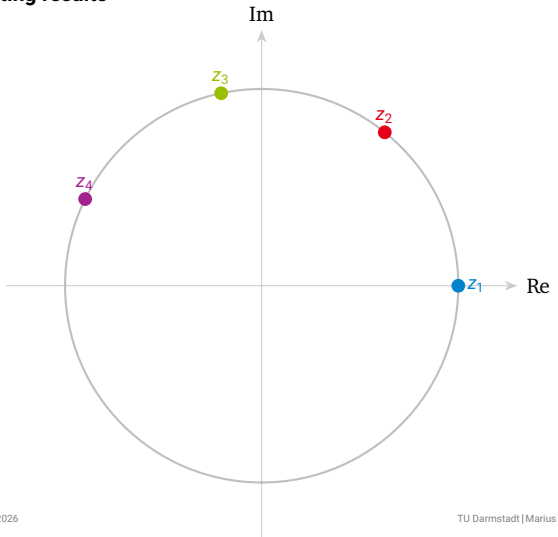
Any 4 columns of matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & z_2^2 & z_3^2 & z_4^2 & z_5^2 \\ 1 & z_2^5 & z_3^5 & z_4^5 & z_5^5 \\ 1 & z_2^7 & z_3^7 & z_4^7 & z_5^7 \end{bmatrix}$$

are linearly dependent.

NON-UNIFORM AMBIGUITY SET 4

Existing results



Inter-element spacing: $0.6\,\lambda/2$

Ambiguous spatial frequencies:

$$\arg z_1 = 0, \quad \arg z_2 = 0.8944,$$

$$\arg z_3 = 1.7783, \quad \arg z_4 = 2.6853,$$

The columns of matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & z_2^2 & z_3^2 & z_4^2 \\ 1 & z_2^5 & z_3^5 & z_4^5 \\ 1 & z_2^7 & z_3^7 & z_4^7 \end{bmatrix}$$

are linearly dependent.

NON-UNIFORM AMBIGUITIES

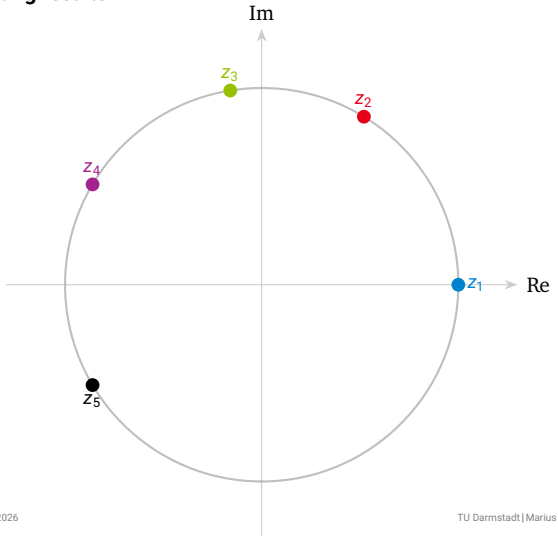
Existing results

Why does the reversed procedure work?

- The number of distinct ambiguity sets and nullspace dimensions are sufficiently large (as we will later see).
- The constructed polynomial has centro-Hermitian coefficients, which is a necessary condition for constructing polynomials with all roots on the unit-circle.
- The authors themselves have recognized that:
  - a sufficiently large number of roots on the unit-circle may not be obtained for the constructed polnymials.
  - the ambiguity characterization is not exhaustive, i.e. many other ambiguity generator sets exist.

NON-UNIFORM AMBIGUITY SET 1

Existing results



Inter-element spacing: $0.6\,\lambda/2$

Ambiguous spatial frequencies:

$$\arg z_1 = 0, \quad \arg z_2 = 1.0241,$$

$$\arg z_3 = 1.7312, \quad \arg z_4 = 2.6064,$$

$$\arg z_5 = -2.6064,$$

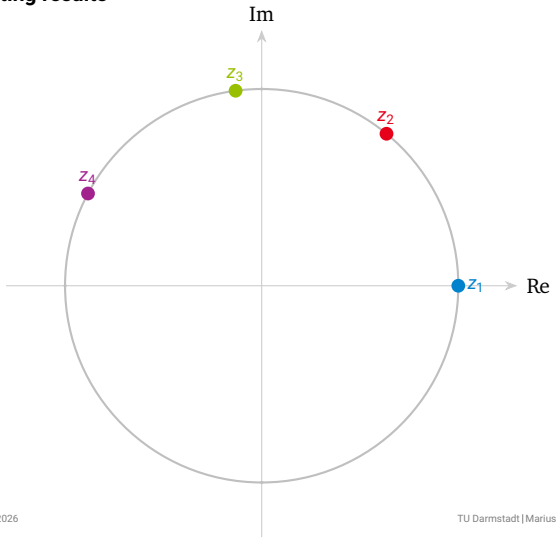
Any 4 columns of matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & z_2^2 & z_3^2 & z_4^2 & z_5^2 \\ 1 & z_2^5 & z_3^5 & z_4^5 & z_5^5 \\ 1 & z_2^7 & z_3^7 & z_4^7 & z_5^7 \end{bmatrix}$$

are linearly dependent.

NON-UNIFORM AMBIGUITY SET 2

Existing results



Inter-element spacing: $0.6\,\lambda/2$

Ambiguous spatial frequencies:

$$\arg z_1 = 0, \quad \arg z_2 = 0.8830,$$

$$\arg z_3 = 1.7038, \quad \arg z_4 = 2.6536,$$

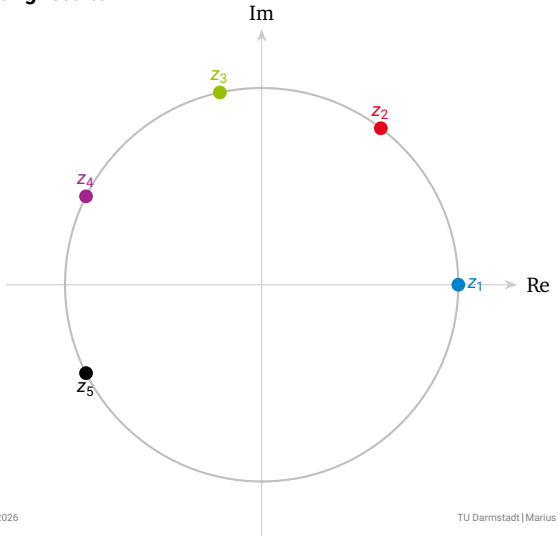
The columns of matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & z_2^2 & z_3^2 & z_4^2 \\ 1 & z_2^5 & z_3^5 & z_4^5 \\ 1 & z_2^7 & z_3^7 & z_4^7 \end{bmatrix}$$

are linearly dependent.

NON-UNIFORM AMBIGUITY SET 3

Existing results



Inter-element spacing: $0.6\,\lambda/2$

Ambiguous spatial frequencies:

$$\arg z_1 = 0, \quad \arg z_2 = 0.9197,$$

$$\arg z_3 = 1.7851, \quad \arg z_4 = 2.6756,$$

$$\arg z_5 = -2.6756,$$

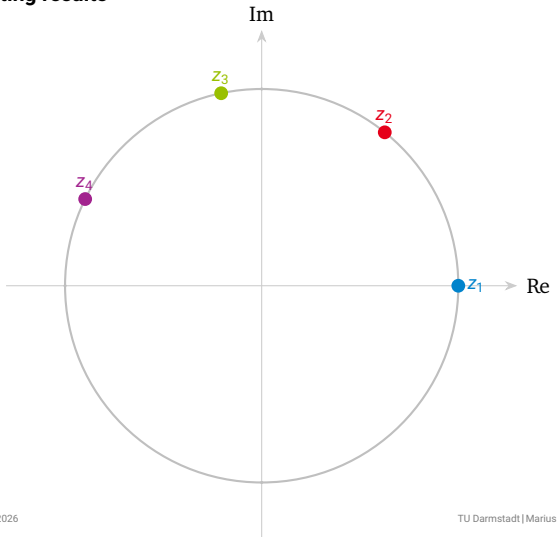
Any 4 columns of matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & z_2^2 & z_3^2 & z_4^2 & z_5^2 \\ 1 & z_2^5 & z_3^5 & z_4^5 & z_5^5 \\ 1 & z_2^7 & z_3^7 & z_4^7 & z_5^7 \end{bmatrix}$$

are linearly dependent.

NON-UNIFORM AMBIGUITY SET 4

Existing results



Inter-element spacing: $0.6\,\lambda/2$

Ambiguous spatial frequencies:

$$\arg z_1 = 0, \quad \arg z_2 = 0.8944,$$

$$\arg z_3 = 1.7783, \quad \arg z_4 = 2.6853,$$

The columns of matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & z_2^2 & z_3^2 & z_4^2 \\ 1 & z_2^5 & z_3^5 & z_4^5 \\ 1 & z_2^7 & z_3^7 & z_4^7 \end{bmatrix}$$

are linearly dependent.

SPARSE ARRAY ANALYSIS: SCHUR POLYNOMIAL APPROACH

Existing results

In Matter et al[11], we take an alternative approach based on:

- The construction of a Schur polynomial based on the Young tableaux and the analytic factorization to determine the roots $z_1, \ldots, z_L$.
- The formulation of a mixed-integer program under the roots of unity assumption (discretized root locations on the unit-circle).

Let $\boldsymbol{A}_{\mathcal{I}}(z_1, \ldots, z_{M_{\mathcal{I}}})$ denote a $M_{\mathcal{I}} \times M_{\mathcal{I}}$ generalized Vandermonde matrix. Similarly, let $\boldsymbol{A}_{M_{\mathcal{I}}}(z_1, \ldots, z_{M_{\mathcal{I}}})$ denote a $M_{\mathcal{I}} \times M_{\mathcal{I}}$ (classical) Vandermonde matrix.
Furthermore, let

$$V_{M_{\mathcal{I}}}(z_1, \ldots, z_{M_{\mathcal{I}}}) = \det\left(\boldsymbol{A}_{M_{\mathcal{I}}}(z_1, \ldots, z_{M_{\mathcal{I}}})\right) = \prod_{1 \le i < k \le M_{\mathcal{I}}} (z_k - z_i)$$

denote the classical Vandermonde determinant (which does not vanish as long as the generators $z_i \neq z_k$ for all pairs k, i).

[11] F. Matter et al., "Ambiguities in DoA Estimation With Linear Arrays", 2022

SPARSE ARRAY ANALYSIS: SCHUR POLYNOMIAL APPROACH

Existing results

Define

$$V_{\mathcal{I}}(z_1, \ldots, z_{M_{\mathcal{I}}}) = \det\left(\boldsymbol{A}_{\mathcal{I}}(z_1, \ldots, z_{M_{\mathcal{I}}})\right)$$

as the generalized Vandermonde determinant.
Then it is well known[12] that $V_{M_{\mathcal{I}}}(z_1, \ldots, z_{M_{\mathcal{I}}})$ divides $V_{\mathcal{I}}(z_1, \ldots, z_{M_{\mathcal{I}}})$.

The ratio

$$S_{\mathcal{I}}(z_1, \ldots, z_{M_{\mathcal{I}}}) = \frac{V_{\mathcal{I}}(z_1, \ldots, z_{M_{\mathcal{I}}})}{V_{M_{\mathcal{I}}}(z_1, \ldots, z_{M_{\mathcal{I}}})} = \sum_{R \in R_{\mathcal{I}}} \prod_{i=1}^{M_{\mathcal{I}}} z_i^{\alpha_i(R)}$$

is the so-called Schur polynomial that must vanish if $\boldsymbol{A}_{\mathcal{I}}(z_1, \ldots, z_{M_{\mathcal{I}}})$ drops rank, i.e., $S_{\mathcal{I}}(z_1, \ldots, z_{M_{\mathcal{I}}}) = 0$.

Here $R_{\mathcal{I}}$ is the set of all Standard Young tableaux with entries in $\{1, \ldots, M\}$, and $\alpha_i(R)$ denotes integers that can be determined from the corresponding Young tableaux[11].

[12] S. De Marchi, "Polynomials arising in factoring generalized Vandermonde determinants: an algorithm for computing their coefficients", 2001

[11] F. Matter et al., "Ambiguities in DoA Estimation With Linear Arrays", 2022

SPARSE ARRAY ANALYSIS: SCHUR POLYNOMIAL APPROACH

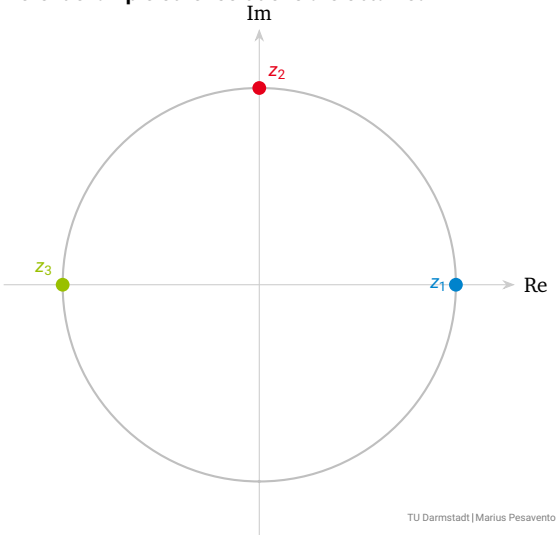
Existing results

For example for 3 sensor sparse (thinned) array obtained from a 5 sensor ULA the Schur polynomial becomes

$$S_{\mathcal{I}}(z_1, \ldots, z_{M_{\mathcal{I}}}) = z_1 z_2 z_3 \, (z_1^2 + z_1 z_2 + z_2^2 + z_1 z_3 + z_2 z_3 + z_3^2).$$

ALGEBRAIC SOLUTION YIELDS KNOWN SOLUTION FAMILY

but in a different example other solutions are obtained



Uniform generator set:

$$\arg z_1 = 0, \quad \arg z_2 = \tfrac{\pi}{2},$$

$$\arg z_3 = \pi,$$

Singular array steering matrix

$$\boldsymbol{A}(z_1, z_2, z_3) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{\mathrm{j}\frac{\pi}{2}} & e^{\mathrm{j}\pi} \\ 1 & e^{\mathrm{j}4\frac{\pi}{2}} & e^{\mathrm{j}4\pi} \end{bmatrix}$$

are linearly dependent.

MIXED INTEGER APPROACH

Existing results

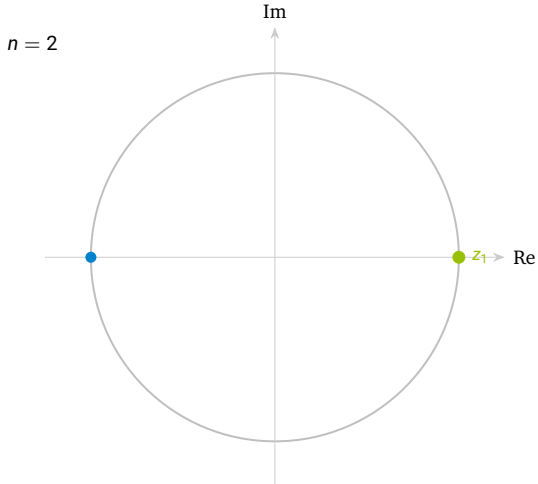
- Determining algebraically all generators $z_1, \ldots, z_L$ on the unit-circle, for which the Schur polynomial vanishes, may not always be possible.
- Alternatively, select solution set from the sets of roots of unity ($z^K = 1$), i.e., the sets

$$\{1, e^{\mathrm{j}\frac{2\pi}{K}}, \ldots, e^{\mathrm{j}\frac{2\pi(K-1)}{K}}\}, \quad (K = 2, \ldots, M)$$

  using a mixed integer programming approach.

TECHNISCHE UNIVERSITÄT DARMSTADT

ROOTS OF UNITY



MIXED INTEGER APPROCH

- This discretization approach is also motivated by the observation that the uniform ambiguities belong to this class.
- This approach has revealed new ambiguity sets that have not been discovered before, however, at a high computational cost.
- The approach does not scale well with the aperture M.

LIMITATIONS OF EXISTING ANALYSIS APPROACHES

- Existing analysis approaches are limited to array steering matrices with the same number of sources as sensors ($L = M_{\mathcal{I}}$), i.e., square generalized Vandermonde matrices.
- In practice, however, we know that all thinned ULAs exhibit ambiguities and we want to determine all ambiguity sets for $L \leq M_{\mathcal{I}}$ or prove that for a given source number L the array is free of ambiguities.
- Existing analysis approaches do not fully characterize all ambiguity sets. In fact, it is known, that each approach is only characterizing particular subsets of ambiguities.
- Existing analysis approaches do not scale well with the aperture M and the number of sensors $M_{\mathcal{I}}$ in the sparse array, or the omitted sensors $M_{\bar{\mathcal{I}}}$.
- Existing ambiguity analysis approaches are not constructive in the sense that they allow to draw conclusions on the design of sparse arrays that are free from ambiguities.

Sparse array analysis

Generalized Vandermonde rank conditions

SCALABLE AMBIGUITY CHARACTERIZATION APPROACH

for thinned uniform linear arrays

- We consider an arbitrary source number $L < M_{\mathcal{I}}$, i.e., the general tall rather than the square steering matrix case:

$$\boldsymbol{y}_{\mathcal{I}} = \boldsymbol{A}_{\mathcal{I}}(z_1, \ldots, z_L)\,\boldsymbol{x} \quad + \text{noise}$$

- $\boldsymbol{A}_{\mathcal{I}}(z_1, \ldots, z_L)$ is a generalized Vandermonde obtained from thinning a full Vandermonde:

$$\boldsymbol{A}_{\mathcal{I}}(z_1, z_2, z_3 \cdots, z_5) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ z_1 & z_2 & z_3 & z_4 & z_5 \\ z_1^2 & z_2^2 & z_3^2 & z_4^2 & z_5^2 \\ z_1^3 & z_2^3 & z_3^3 & z_4^3 & z_5^3 \\ z_1^4 & z_2^4 & z_3^4 & z_4^4 & z_5^4 \\ z_1^5 & z_2^5 & z_3^5 & z_4^5 & z_5^5 \\ z_1^6 & z_2^6 & z_3^6 & z_4^6 & z_5^6 \\ z_1^7 & z_2^7 & z_3^7 & z_4^7 & z_5^7 \\ z_1^8 & z_2^8 & z_3^8 & z_4^8 & z_5^8 \end{bmatrix}.$$

- Generators $z_\ell = e^{\mathrm{j}2\pi\theta_\ell}$ are on the unit-circle and $\theta_\ell = \frac{2\pi d}{\lambda}\cos\vartheta$ are the electrical angles (direction cosines) corresponding to the impinging directions ϑ.

THINNED POLYNOMIAL EXAMPLE: $L = 5$

Fundamental questions:

1. What is the maximum number of sources L, for which $\boldsymbol{A}_{\mathcal{I}}(z_1, z_2, z_3 \cdots, z_L)$ is always full rank? This corresponds to determining the Kruskal rank = spark $\left(\boldsymbol{A}_{\mathcal{I}}(z_1, z_2, z_3 \cdots, z_L)\right) - 1$.
2. For fixed number L, what are the generator sets $\{z_1, \ldots, z_L\}$ for which $\boldsymbol{A}_{\mathcal{I}}(z_1, z_2, z_3 \cdots, z_L)$ becomes rank deficient? The corresponding scenarios are not resolvable!

To answer these questions define the Toeplitz matrix $\mathcal{T}(\boldsymbol{g})$ and the complementary submatrix $\mathcal{T}_{\bar{\mathcal{I}}}(\boldsymbol{g})$:

$$\mathcal{T}(\boldsymbol{g}) = \begin{bmatrix} g_0 & g_1 & g_2 & g_2^* & g_1^* & g_0^* & 0 & 0 & 0 \\ 0 & g_0 & g_1 & g_2 & g_2^* & g_1^* & g_0^* & 0 & 0 \\ 0 & 0 & g_0 & g_1 & g_2 & g_2^* & g_1^* & g_0^* & 0 \\ 0 & 0 & 0 & g_0 & g_1 & g_2 & g_2^* & g_1^* & g_0^* \end{bmatrix} \Rightarrow \quad \mathcal{T}_{\bar{\mathcal{I}}}(\boldsymbol{g}) = \begin{bmatrix} g_1 & g_2^* & 0 \\ g_0 & g_2 & 0 \\ 0 & g_1 & g_0^* \\ 0 & g_0 & g_1^* \end{bmatrix}$$

MAIN RESULT

For the complete proof see pre-print: http://arxiv.org/abs/2606.00360

Theorem 1

For the thinned array given above with $M = 9$, $L = 5$ and the sensor set $\mathcal{I} = \{1, 3, 5, 6, 7, 9\}$, the coefficient vector $\boldsymbol{g} = [g_0, g_1, g_2, g_2^*, g_1^*, g_0^*]^{\mathsf{T}}$, with bounding coefficient $g_0 \neq 0$, for which the complementary submatrix

$$\mathcal{T}_{\bar{\mathcal{I}}}(\boldsymbol{g}) = \begin{bmatrix} g_1 & g_2^* & 0 \\ g_0 & g_2 & 0 \\ 0 & g_1 & g_0^* \\ 0 & g_0 & g_1^* \end{bmatrix}$$

becomes rank deficient, yields a full polynomial

$$P_{\boldsymbol{g}}(t) = g_0 + g_1 z + g_2 z^2 + g_2^* z^3 + g_1^* z^4 + g_0^* z^5 = k \prod_{\ell=1}^{5} (z - z_\ell).$$

with roots $z_1, \ldots, z_5$ and constant k, for which $\boldsymbol{A}_{\mathcal{I}}(z_1, z_2, z_3 \cdots, z_5)$ becomes rank deficient (if all roots are distinct and on the unit-circle).

SPARSE ARRAY EXAMPLE

- Setting $g_0 = 1$ and performing Gaussian elimination with pivoting yields

$$\mathcal{T}_{\bar{\mathcal{I}}}(\boldsymbol{g}) = \begin{bmatrix} g_1 & g_2^* & 0 \\ g_0 & g_2 & 0 \\ 0 & g_1 & g_0^* \\ 0 & g_0 & g_1^* \end{bmatrix} \overset{\text{row}}{\equiv} \begin{bmatrix} 1 & g_2 & 0 \\ 0 & 1 & g_1^* \\ 0 & 0 & g_1^*|g_2|(e^{\mathrm{j}\phi_2} g_1 - e^{-\mathrm{j}\phi_2}) \\ 0 & 0 & 1 - |g_1|^2 \end{bmatrix}.$$

which is rank deficient for:

1. $g_2 = 0$ and $|g_1| = 1$:

$$P_{\boldsymbol{g}}(z) = 1 + z + z^4 + z^5 = (z+1) \prod_{p=1}^{4} \left(z - e^{\mathrm{j}(2p-1)\pi/4}\right).$$

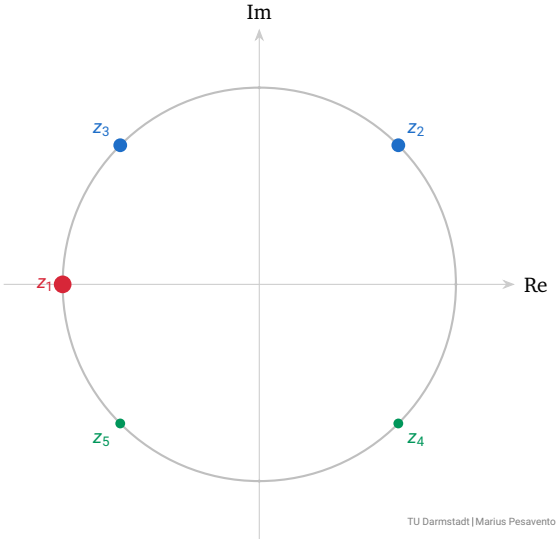
2. $g_2 = |g_2| e^{\mathrm{j}\phi_2} \neq 0$ and $g_1 = e^{-\mathrm{j}2\phi_2}$:

$$P_{\boldsymbol{g}}(z) = \left(z e^{-\mathrm{j}2\phi_2} + 1\right)\left(e^{\mathrm{j}\phi_2} z^2 + \alpha\right)\left(e^{\mathrm{j}\phi_2} z^2 + \alpha^*\right)$$

- By Theorem 1 the array steering matrix $\boldsymbol{A}_{\mathcal{I}}(z_1, \ldots, z_5) \in \mathbb{C}^{7\times 5}$ is rank deficient.

TECHNISCHE UNIVERSITÄT DARMSTADT

SOLUTION 1: AMBIGUOUS GENERATOR SET



Fixed root locations

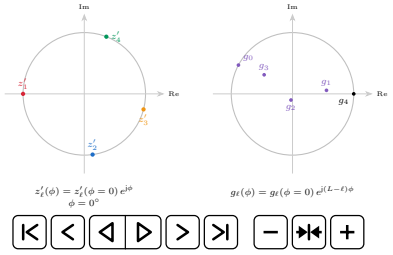
$$\arg z_1 = -\pi,$$

$$\arg z_{2,3} = \tfrac{\pi}{4},\ \tfrac{3\pi}{4},$$

$$\arg z_{4,5} = \tfrac{7\pi}{4},\ \tfrac{5\pi}{4}.$$

All five lie on the unit-circle.

SOLUTION 2: AMBIGUOUS GENERATOR SETS



The five roots

$$z_1 = -e^{j2\phi_2}, \quad z_{2,3} = \pm j\, e^{j(\psi-\phi_2)/2},$$

$$z_{4,5} = \pm j\, e^{-j(\psi+\phi_2)/2}$$

all lie on the unit-circle, with

$$\phi_2 \in [-\pi, \pi], \qquad \psi = \arccos\big(|g_2|/2\big),$$

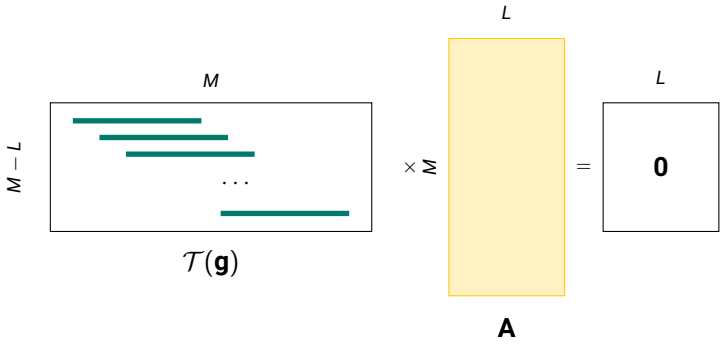
and $|g_2| \in [0, 2]$.

TECHNISCHE UNIVERSITÄT DARMSTADT

USE A CONVENIENT TOEPLITZ MATRIX REPRESENTATION

Sketch of proof of Theorem 1. For details see: http://arxiv.org/abs/2606.00360

For full Vandermonde matrix ***A***



- Rows of $\mathcal{T}(\mathbf{g})$ built from shifted parameter vector **g**
- These rows form a basis of the orthogonal complement of $\mathcal{R}(\mathbf{A})$, i.e $\mathcal{R}\left(\mathcal{T}^{\mathsf{T}}(\mathbf{g})\right) = \mathcal{R}^{\perp}(\mathbf{A})$

TOEPLITZ STRUCTURE OF $\mathcal{T}(\boldsymbol{g})$

Sketch of proof of Theorem 1. For details see: http://arxiv.org/abs/2606.00360

even: $L{=}2K$

$$\mathcal{T}(\boldsymbol{g}) = \begin{array}{c} \begin{matrix} \scriptstyle 1 & \scriptstyle 2 & \scriptstyle 3 & \scriptstyle 4 & \scriptstyle \cdots & \scriptstyle K-2 & \scriptstyle K-1 & \scriptstyle K & \scriptstyle K+1 & \scriptstyle K+2 & \scriptstyle K+3 & \scriptstyle K+4 & \scriptstyle \cdots & \scriptstyle L-2 & \scriptstyle L-1 & \scriptstyle L & \scriptstyle L+1 & \scriptstyle L+2 & \scriptstyle \cdots & \scriptstyle M \end{matrix} \\ \begin{bmatrix} g_0 & g_1 & g_2 & g_3 & \cdots & g_{K-3} & g_{K-2} & g_{K-1} & g_K & g_{K-1}^* & g_{K-2}^* & g_{K-3}^* & \cdots & g_3^* & g_2^* & g_1^* & g_0^* & 0 & \cdots & 0 \\ 0 & g_0 & g_1 & g_2 & g_3 & \cdots & g_{K-3} & g_{K-2} & g_{K-1} & g_K & g_{K-1}^* & g_{K-2}^* & g_{K-3}^* & \cdots & g_3^* & g_2^* & g_1^* & g_0^* & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & g_0 & g_1 & g_2 & g_3 & \cdots & g_{K-3} & g_{K-2} & g_{K-1} & g_K & g_{K-1}^* & g_{K-2}^* & g_{K-3}^* & \cdots & g_3^* & g_2^* & g_1^* & g_0^* \end{bmatrix} \\ \begin{matrix} \scriptstyle 1 & \scriptstyle \cdots & \scriptstyle M-L-1 & \scriptstyle M-L & \scriptstyle M-L+1 & \scriptstyle M-L+2 & \scriptstyle M-L+3 & \scriptstyle \cdots & \scriptstyle M-K-2 & \scriptstyle M-K-1 & \scriptstyle M-K & \scriptstyle M-K+1 & \scriptstyle M-K+2 & \scriptstyle M-K+3 & \scriptstyle M-K+4 & \scriptstyle \cdots & \scriptstyle M-3 & \scriptstyle M-2 & \scriptstyle M-1 & \scriptstyle M \end{matrix} \end{array}$$

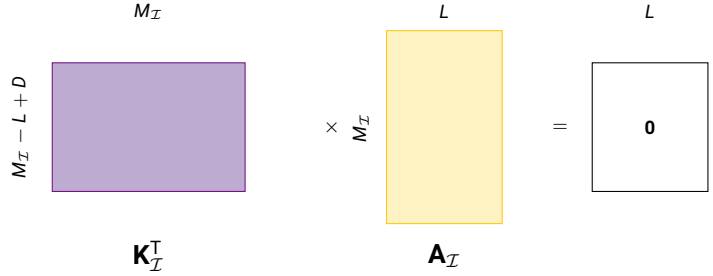
odd: $L{=}2K{+}1$

$$\mathcal{T}(\boldsymbol{g}) = \begin{array}{c} \begin{matrix} \scriptstyle 1 & \scriptstyle 2 & \scriptstyle 3 & \scriptstyle 4 & \scriptstyle \cdots & \scriptstyle K-2 & \scriptstyle K-1 & \scriptstyle K & \scriptstyle K+1 & \scriptstyle K+2 & \scriptstyle K+3 & \scriptstyle K+4 & \scriptstyle \cdots & \scriptstyle L-2 & \scriptstyle L-1 & \scriptstyle L & \scriptstyle L+1 & \scriptstyle L+2 & \scriptstyle \cdots & \scriptstyle M \end{matrix} \\ \begin{bmatrix} g_0 & g_1 & g_2 & g_3 & \cdots & g_{K-3} & g_{K-2} & g_{K-1} & g_K & g_K^* & g_{K-1}^* & g_{K-2}^* & \cdots & g_3^* & g_2^* & g_1^* & g_0^* & 0 & \cdots & 0 \\ 0 & g_0 & g_1 & g_2 & g_3 & \cdots & g_{K-3} & g_{K-2} & g_{K-1} & g_K & g_K^* & g_{K-1}^* & g_{K-2}^* & \cdots & g_3^* & g_2^* & g_1^* & g_0^* & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & g_0 & g_1 & g_2 & g_3 & \cdots & g_{K-3} & g_{K-2} & g_{K-1} & g_K & g_K^* & g_{K-1}^* & g_{K-2}^* & \cdots & g_3^* & g_2^* & g_1^* & g_0^* \end{bmatrix} \\ \begin{matrix} \scriptstyle 1 & \scriptstyle \cdots & \scriptstyle M-L-1 & \scriptstyle M-L & \scriptstyle M-L+1 & \scriptstyle M-L+2 & \scriptstyle M-L+3 & \scriptstyle \cdots & \scriptstyle M-K-3 & \scriptstyle M-K-2 & \scriptstyle M-K-1 & \scriptstyle M-K & \scriptstyle M-K+1 & \scriptstyle M-K+2 & \scriptstyle M-K+3 & \scriptstyle \cdots & \scriptstyle M-3 & \scriptstyle M-2 & \scriptstyle M-1 & \scriptstyle M \end{matrix} \end{array}$$

TECHNISCHE UNIVERSITÄT DARMSTADT

THINNED VANDERMONDE $\mathbf{A}_{\mathcal{I}}$ WITH A RANK DEFICIENCY OF D

Sketch of proof of Theorem 1. For details see: http://arxiv.org/abs/2606.00360



$\mathbf{A}_{\mathcal{I}}$ has a rank deficiency of D

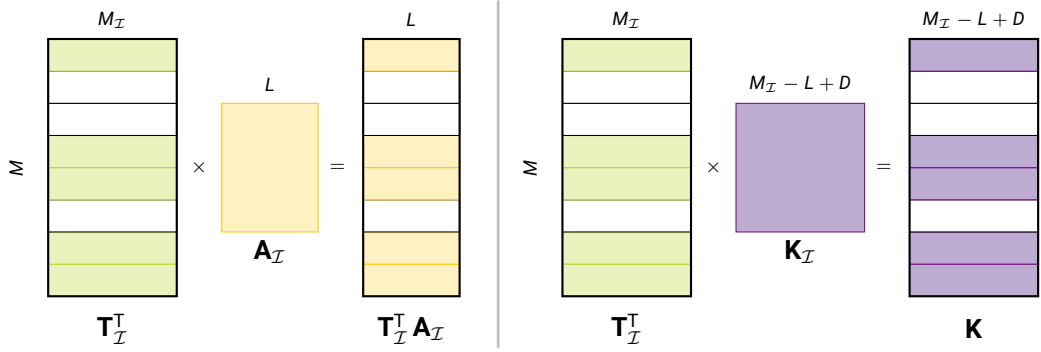
$\Updownarrow$

There exists a full column rank matrix $\mathbf{K}_{\mathcal{I}}$ in orthogonal range space $\mathcal{R}^{\perp}\left(\mathbf{A}_{\mathcal{I}}\right)$.

TECHNISCHE UNIVERSITÄT DARMSTADT

INSERTING ZERO ROWS VIA SELECTION MATRIX $\mathbf{T}_{\mathcal{I}}^{\mathsf{T}}$

Sketch of proof of Theorem 1. For details see: http://arxiv.org/abs/2606.00360



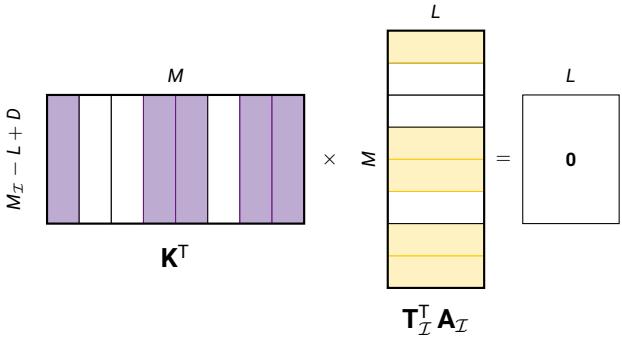
The same selector $\mathcal{T}_{\mathcal{I}}^{\mathsf{T}}$ inserts zero rows at $\bar{\mathcal{I}}$ into both $\mathbf{A}_{\mathcal{I}} \to \mathbf{A}$ and $\mathbf{K}_{\mathcal{I}} \to \mathbf{K}$

This allows for defining a matrix $\mathbf{K} = \mathbf{T}_{\mathcal{I}}^{\mathsf{T}}\mathbf{K}_{\mathcal{I}}$ such that $\mathbf{K}^{\mathsf{T}}\mathbf{A} = \mathbf{0}$!

TECHNISCHE UNIVERSITÄT DARMSTADT

INDUCED ROW SPARSITY PRESERVES NULLSPACE

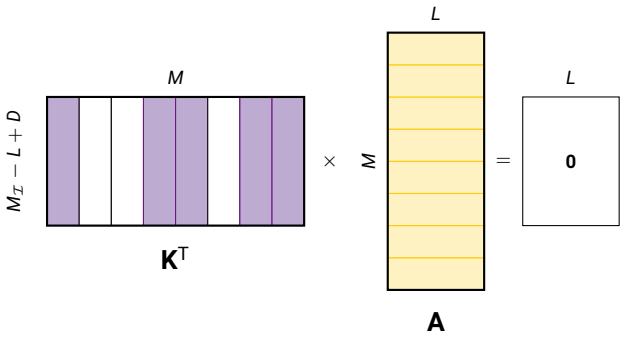
Sketch of proof of Theorem 1. For details see: http://arxiv.org/abs/2606.00360



TECHNISCHE UNIVERSITÄT DARMSTADT

INDUCED ROW SPARSITY PRESERVES NULLSPACE

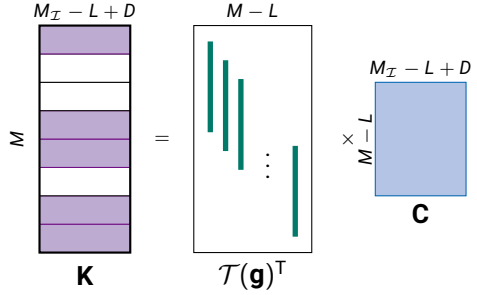
Sketch of proof of Theorem 1. For details see: http://arxiv.org/abs/2606.00360



$\mathbf{K}^{\mathsf{T}}\mathbf{A} = \mathbf{0}$, with zero columns of $\mathbf{K}^{\mathsf{T}}$ and zero rows of $\mathbf{A}$ at the same positions.$\Rightarrow \mathcal{R}(\boldsymbol{K}) \subset \mathcal{R}^{\perp}(\mathbf{A})$

K LIVES IN THE ROW SPACE OF $\mathcal{T}(\mathbf{g})$

Sketch of proof of Theorem 1. For details see: http://arxiv.org/abs/2606.00360



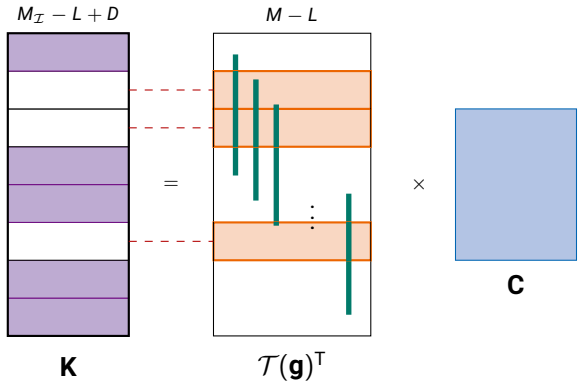
$$\mathcal{R}(\boldsymbol{K}) \subset \mathcal{R}\big(\mathcal{T}(\mathbf{g})^{\mathsf{T}}\big) = \mathcal{R}^{\perp}(\boldsymbol{A})$$

$$\Updownarrow$$

There exists a full column rank $\mathbf{C}$ such that $\mathbf{K} = \mathcal{T}(\mathbf{g})^{\mathsf{T}}\mathbf{C}$.

MARKING THE ROWS $\bar{\mathcal{I}}$ ON BOTH SIDES

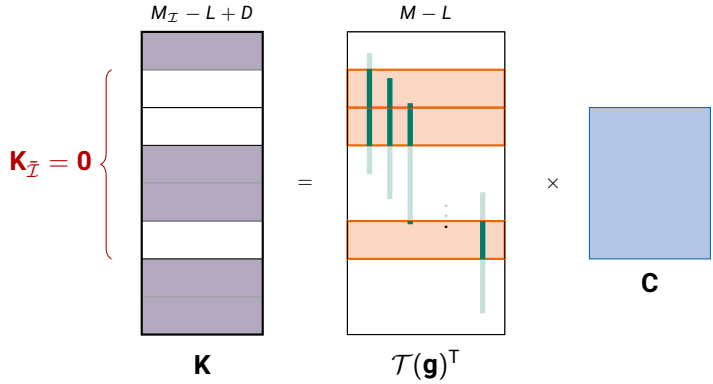
Sketch of proof of Theorem 1. For details see: http://arxiv.org/abs/2606.00360



The rows belonging to $\bar{\mathcal{I}}$ (dashed) occupy the *same positions* in **K** and in $\mathcal{T}(\mathbf{g})^{\mathsf{T}}$.

EXTRACTING THOSE ROWS YIELDS THE KEY IDENTITY

Sketch of proof of Theorem 1. For details see: http://arxiv.org/abs/2606.00360

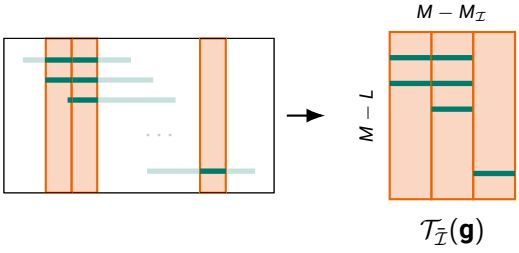


Only the marked rows matter here; the rest of **K** and $\mathcal{T}(\mathbf{g})^{\mathsf{T}}$ (greyed out) is irrelevant

TECHNISCHE UNIVERSITÄT DARMSTADT

FUNDAMENTAL RANK-DEFICIENCY CONDITION

Sketch of proof of Theorem 1. For details see: http://arxiv.org/abs/2606.00360



$\mathcal{T}_{\bar{\mathcal{I}}}(\mathbf{g})^{\mathsf{T}}\mathbf{C} = \mathbf{0}$
with $\mathbf{C}$ full column rank

$\Longrightarrow$ $\mathcal{T}_{\bar{\mathcal{I}}}(\mathbf{g})$ **must be rank deficient**

$\Updownarrow$

$\mathbf{A}_{\mathcal{I}}(z_1, \ldots, z_L)$ **is rank deficient**

MAIN RESULT

For the complete proof see pre-print: http://arxiv.org/abs/2606.00360

Theorem 1 (Rank Condition for Generalized Vandermonde Matrices)

Let $\boldsymbol{A}_{\mathcal{I}}(z_1, \ldots, z_L) \in \mathbb{C}^{M_{\mathcal{I}} \times L}$ *be a generalized Vandermonde matrix with index set* $\mathcal{I}$ *and* $L \leq M_{\mathcal{I}} = |\mathcal{I}|$*, and let* $\boldsymbol{g} = [g_0, \ldots, g_L]^{\mathsf{T}}$ *be the coefficient vector of the centro-Hermitian polynomial* $P_{\boldsymbol{g}}(z) = \sum_{\ell=0}^{L} g_\ell z^\ell$ *whose roots are* $z_1, \ldots, z_L$*. Then* $\boldsymbol{A}_{\mathcal{I}}(z_1, \ldots, z_L)$ *is rank deficient if and only if the complementary submatrix*

$$\mathcal{T}_{\bar{\mathcal{I}}}(\boldsymbol{g}) = \mathcal{T}(\boldsymbol{g})\, \boldsymbol{T}_{\bar{\mathcal{I}}}^{\mathsf{T}} \in \mathbb{C}^{(M-L)\times(M-M_{\mathcal{I}})}$$

is rank deficient, i.e.,

$$\operatorname{rank}\big(\mathcal{T}_{\bar{\mathcal{I}}}(\boldsymbol{g})\big) = M - M_{\mathcal{I}} - D$$

for some integer D *with* $1 \leq D < L$*.*

RANK INVARIANCE WITH RESPECT TO ROOT ROTATION

- The array polynomial is built directly from its L roots, all lying on the unit-circle:

$$P(z) \;=\; \prod_{\ell=1}^{L}(z - z'_\ell) \;=\; \sum_{\ell=0}^{L} g_\ell \, z^\ell, \qquad g_L = 1$$

- Each g_ℓ is an elementary symmetric function $\varepsilon_k(z'_1, \ldots, z'_L) = \sum_{1 \le i_1 < \cdots < i_k \le L} z'_{i_1} \cdots z'_{i_k}$. of the roots

$$g_\ell = (-1)^{L-\ell}\, \varepsilon_{L-\ell}(z'_1, \ldots, z'_L)$$

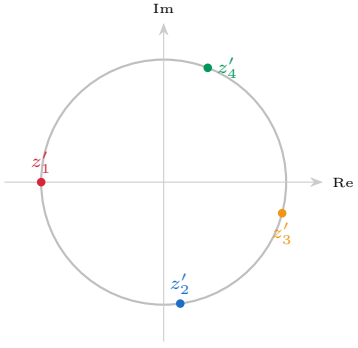
- Rotation of every root by the same shift ϕ: $\quad z'_\ell \;\longrightarrow\; z'_\ell \, e^{j\phi}$
- Since ε_k contains sums of exactly k root factors, e.g., $z'_{\ell_1} z'_{\ell_2}, \ldots, z'_{\ell_k}$, each coefficient inherits a phase:

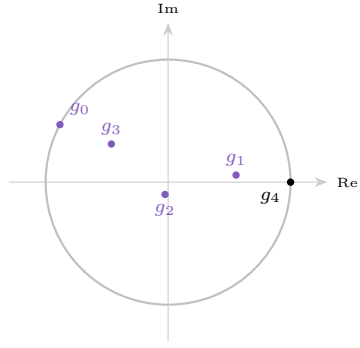
$$g_\ell(\phi) \;=\; g_\ell(\phi = 0)\, e^{j(L-\ell)\phi}$$

- A common rotation of the *roots* becomes a linear phase increment across the coefficient index: the phase increment is absorbed in the polynomial coefficients.

TECHNISCHE UNIVERSITÄT DARMSTADT

ROTATING ROOTS VS PLAIN COEFFICIENTS





$$z'_\ell(\phi) = z'_\ell(\phi = 0)\, e^{\mathrm{j}\phi}$$

$$\phi = 0^\circ$$

$$g_\ell(\phi) = g_\ell(\phi = 0)\, e^{\mathrm{j}(L-\ell)\phi}$$

RESTORING CONJUGATE SYMMETRY UNDER ROTATION

- The polynomials from the roots are self-inversive (*Geometry of Polynomials*, 1966)[13]
  1. Definition: $g(z) = \sum_\ell b_l z^\ell$ is self-inversive if

$$b_{L-\ell} = u\, b_\ell^*, \qquad |u| = 1.$$

  2. Roots on unit-circle ($|z| = 1$) $\Rightarrow$ self-inversive: each root is its own mirror image under $z \mapsto 1/z^*$.
  3. Self-inversive $\Rightarrow$ conjugate-symmetric coefficients, up to u:
     Write $u = e^{-2\mathrm{j}\alpha}$; rotating by $e^{\mathrm{j}\alpha}$ gives $h_\ell = b_\ell e^{\mathrm{j}\alpha}$ with $h_{L-\ell} = h_\ell^*$.

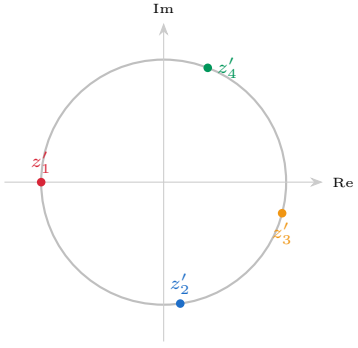
- Rotating the roots ($z_\ell \to z_\ell e^{\mathrm{j}\phi}$) keeps $|z_\ell| = 1$, so u is unchanged. Counter-rotating by $e^{-\mathrm{j}L\phi/2}$:

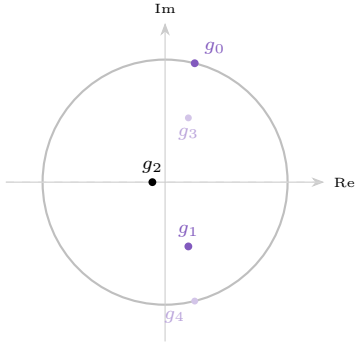
$$g_\ell(\phi) = g_\ell(0)\, e^{\mathrm{j}\alpha}\, e^{\mathrm{j}(\frac{L}{2}-\ell)\phi} \quad \Longrightarrow \quad g_{L-\ell}(\phi) = g_\ell^*(\phi) \;\; \forall\, \phi.$$

[13] M. Marden, "Geometry of Polynomials", 1966

TECHNISCHE UNIVERSITÄT DARMSTADT

ROTATING ROOTS VS COEFFICIENTS PRESERVING SYMMETRY





$$z'_\ell(\phi) = z'_\ell(\phi = 0)\, e^{\mathrm{j}\phi}$$
$$\phi = 0^\circ$$

$$g_\ell(\phi) = g_\ell(\phi = 0) e^{\mathrm{j}\alpha} e^{\mathrm{j}(\frac{L}{2} - \ell)\phi}$$

ROTATION INVARIANCE IN COMPLEMENTARY SUBMATRIX

Corollary 1 in pre-print http://arxiv.org/abs/2606.00360

It is easy to show that

Corollary 1

The rotation invariance property is inherited by the complementary submatrix

$$\operatorname{rank}\big(\mathcal{T}_{\bar{\mathcal{I}}}(\boldsymbol{g})\big) = \operatorname{rank}\big(\mathcal{T}_{\bar{\mathcal{I}}}(\boldsymbol{g}(\phi))\big).$$

CONJUGATE INVARIANCE OF AMBIGUOUS GENERATOR SETS

Corollary 2 in pre-print http://arxiv.org/abs/2606.00360

Corollary 2

The complementary submatrix $\mathcal{T}_{\bar{\mathcal{I}}}(\boldsymbol{g})$ is rank deficient if and only if $\mathcal{T}_{\bar{\mathcal{I}}}(\boldsymbol{g}^*)$ is rank deficient.

Hence, taking complex conjugates of all the coefficients g_ℓ does not change whether the associated Toeplitz submatrix loses rank.

This implies that if $\{z_1, \ldots, z_L\}$ are the roots of $P_{\boldsymbol{g}}(z)$, then $\{z_1^*, \ldots, z_L^*\}$ are the roots of $P_{\boldsymbol{g}^*}(z)$, since

$$P_{\boldsymbol{g}^*}(z) = \big(P_{\boldsymbol{g}}(z^*)\big)^*.$$

CONJUGATING THE COMPLEMENTARY SUBMATRIX

Sketch of proof of Corollary 2: see pre-print http://arxiv.org/abs/2606.00360

Inspection of the definitions of $\mathcal{T}(\boldsymbol{g})$ (even and odd cases) shows that replacing each coefficient g_ℓ by g_ℓ^* replaces *every entry* of $\mathcal{T}(\boldsymbol{g})$ by its complex conjugate:

$$\mathcal{T}(\boldsymbol{g}^*) = \mathcal{T}(\boldsymbol{g})^*.$$

Since $\boldsymbol{T}_{\bar{\mathcal{I}}}$ is a *real* selection matrix, conjugation passes through it:

$$\mathcal{T}_{\bar{\mathcal{I}}}(\boldsymbol{g}^*) = \mathcal{T}(\boldsymbol{g}^*)\,\boldsymbol{T}_{\bar{\mathcal{I}}}^{\mathsf{T}} = \left(\mathcal{T}(\boldsymbol{g})\,\boldsymbol{T}_{\bar{\mathcal{I}}}^{\mathsf{T}}\right)^* = \mathcal{T}_{\bar{\mathcal{I}}}(\boldsymbol{g})^*.$$

CONCLUSION

Sketch of proof of Corollary 2: see pre-print http://arxiv.org/abs/2606.00360

Since the rank of a matrix is invariant under complex conjugation,

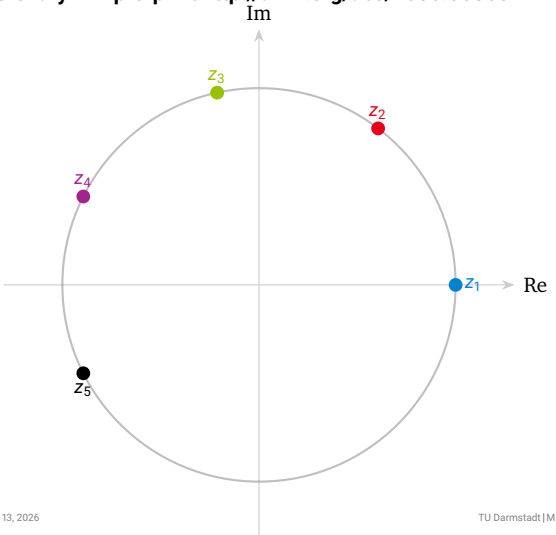
$$\operatorname{rank}\left(\mathcal{T}_{\bar{\mathcal{I}}}(\boldsymbol{g}^*)\right) = \operatorname{rank}\left(\mathcal{T}_{\bar{\mathcal{I}}}(\boldsymbol{g})^*\right) = \operatorname{rank}\left(\mathcal{T}_{\bar{\mathcal{I}}}(\boldsymbol{g})\right).$$

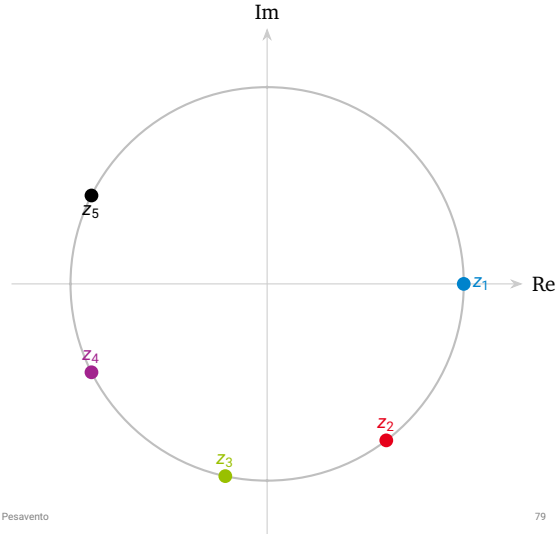
Hence $\mathcal{T}_{\bar{\mathcal{I}}}(\boldsymbol{g})$ is rank deficient if and only if $\mathcal{T}_{\bar{\mathcal{I}}}(\boldsymbol{g}^*)$ is.

TECHNISCHE UNIVERSITÄT DARMSTADT

CONJUGATE INVARIANCE OF AMBIGUOUS GENERATOR SETS

Corollary 2 in pre-print http://arxiv.org/abs/2606.00360





VIOLATION OF SPATIAL SAMPLING THEOREM

Corollary 3 in pre-print http://arxiv.org/abs/2606.00360

Corollary 3

For a thinned ULA with $M_{\mathcal{I}} < M$ sensors and a number of sources $L \geq M_{\mathcal{I}}$, the thinned array steering matrix $\boldsymbol{A}_{\mathcal{I}}(z_1, \ldots, z_L)$ is always ambiguous: there exists a set of distinct generators $\{z_1, \ldots, z_L\}$ on the unit-circle for which $\boldsymbol{A}_{\mathcal{I}}(z_1, \ldots, z_L)$ becomes rank deficient.

Sketch of proof

- It suffices to consider $L = M_{\mathcal{I}}$, for which the complementary submatrix $\mathcal{T}_{\bar{\mathcal{I}}}(\boldsymbol{g}) \in \mathbb{C}^{(M-L)\times(M-M_{\mathcal{I}})}$ is $M_{\bar{\mathcal{I}}} \times M_{\bar{\mathcal{I}}}$ square, with $M_{\bar{\mathcal{I}}} = M - M_{\mathcal{I}}$.
- Consider for $\boldsymbol{b} = \boldsymbol{e}_1 + \boldsymbol{e}_M \in \mathbb{B}^M$ the centro-Hermitian polynomial and its factorization

$$P_{\boldsymbol{b}}(z) = 1 + z^{M-1} = \prod_{m=1}^{M-1} \left(z - e^{\mathrm{j}\pi(2m-1)/(M-1)}\right) = \underbrace{\Big(\sum_{\ell=0}^{M_{\bar{\mathcal{I}}}-1} h_\ell\, z^\ell\Big)}_{P_{\boldsymbol{h}}(z)} \underbrace{\Big(\sum_{\ell=0}^{L} g_\ell\, z^\ell\Big)}_{P_{\boldsymbol{g}}(z)}$$

- Hence $P_{\boldsymbol{b}}(z)$ has $M-1$ roots on the unit-circle.

VIOLATION OF SPATIAL SAMPLING THEOREM

Corollary 3 in pre-print http://arxiv.org/abs/2606.00360

- The previous polynomial equation can be written in matrix form as

$$\boldsymbol{b}^{\mathsf{T}} = \underbrace{[1, 0, \ldots, 0, 1]}_{M} = \underbrace{[h_0, h_1, \ldots, h_1^*, h_0^*]}_{M_{\bar{\mathcal{I}}}} \mathcal{T}_{\bar{\mathcal{I}}}$$

- Right-multiplication by the complementary selection matrix $\boldsymbol{T}_{\bar{\mathcal{I}}}^{\mathsf{T}}$ and considering that by defintion $1, M \notin \bar{\mathcal{I}}$ such that $\boldsymbol{b}^{\mathsf{T}} \boldsymbol{T}_{\bar{\mathcal{I}}}^{\mathsf{T}} = \boldsymbol{0}^{\mathsf{T}}$ yields

$$\boldsymbol{b}^{\mathsf{T}} \boldsymbol{T}_{\bar{\mathcal{I}}}^{\mathsf{T}} = \boldsymbol{h}^{\mathsf{T}} \mathcal{T}(\boldsymbol{g}) \boldsymbol{T}_{\bar{\mathcal{I}}}^{\mathsf{T}} = \boldsymbol{h}^{\mathsf{T}} \mathcal{T}_{\bar{\mathcal{I}}}(\boldsymbol{g})$$

$$\Downarrow$$

$$\boldsymbol{0}^{\mathsf{T}} = \boldsymbol{h}^{\mathsf{T}} \mathcal{T}_{\bar{\mathcal{I}}}(\boldsymbol{g})$$

$$\Updownarrow$$

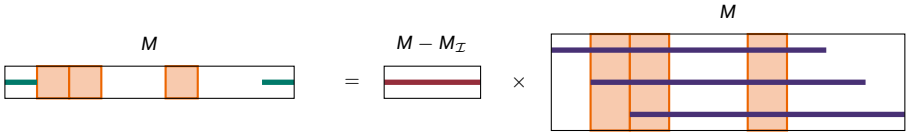
$$\mathcal{T}_{\bar{\mathcal{I}}}(\boldsymbol{g}) \quad \text{rank deficient}$$

- The last equivalence follows from the fact that $\mathcal{T}_{\bar{\mathcal{I}}}(\boldsymbol{g})$ is square and $\boldsymbol{h} \neq \boldsymbol{0}$.

TECHNISCHE UNIVERSITÄT DARMSTADT

VIOLATION OF SPATIAL SAMPLING THEOREM

Corollary 3 in pre-print http://arxiv.org/abs/2606.00360



$\mathbf{b}^{\mathsf{T}} = [1, 0, \ldots, 0, 1]^{\mathsf{T}}$ $\quad \mathbf{h}^{\mathsf{T}} \quad \mathcal{T}(\mathbf{g})$

SPATIAL SAMPLING THEOREM

- Corollary 3 states that, in a thinned ULA, the number of columns L of a thinned steering matrix $\boldsymbol{A}_{\mathcal{I}}(z_1, \ldots, z_L)$ must be strictly less than the number of physical sensors $M_{\mathcal{I}}$, i.e. $L < M_{\mathcal{I}}$.
- This contrasts the full ULA case, where the Vandermonde determinant theorem guarantees full column rank of $\boldsymbol{A}_M(z_1, \ldots, z_L)$ for any $L \leq M$ and any set of distinct generators.
- This result has previously been reported in (Manikas et al,1998) [7] and (Abramovich,1999)[8].

[7] A. Manikas and C. Proukakis, "Modeling and estimation of ambiguities in linear arrays", Aug. 1998

[8] Y. Abramovich et al., "Identifiability and manifold ambiguity in DOA estimation for nonuniform linear antenna arrays", Mar. 1999

SINGLE MISSING SENSOR

Example 1 of pre-print http://arxiv.org/abs/2606.00360

- For $M = 9$ and $L = M_{\mathcal{I}} = M - 1 = 8$ sources, the Toeplitz matrix is the single row

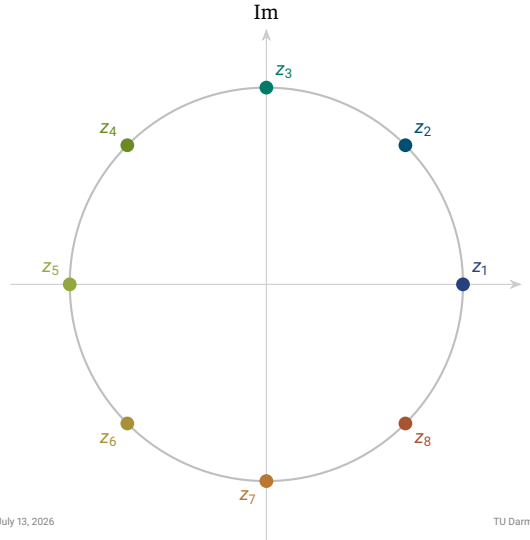
$$\mathcal{T}(\boldsymbol{g}) = [g_0,\, g_1,\, g_2,\, g_3,\, g_4,\, g_3^*,\, g_2^*,\, g_1^*,\, g_0^*],$$

  where $g_0, \ldots, g_3 \in \mathbb{C}$ with $g_0 \neq 0$ and $g_4 \in \mathbb{R}$.
- The complementary submatrix reduces to the scalar $\mathcal{T}_{\bar{\mathcal{I}}}(\boldsymbol{g}) = g_{i-1}$, which vanishes for $g_{i-1} = 0$.
- It can be show that a corresponding polynomial

$$P_{\boldsymbol{g}}(z) = \sum_{\ell=0}^{8} g_\ell z^\ell, \quad g_{i-1} = g_{8-i}^* = 0, \quad i \neq 1, 8$$

  with all roots on the unit-circle always be found.
- By Theorem 1, ambiguities always exist.
- This is a special case of Corollary 1.

ALGEBRAIC SOLUTION BELONGS TO UNIFORM AMBIGUITY FAMILY



$$\boldsymbol{A}_{\mathcal{I}}(z_1,\cdots,z_8) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ z_1 & z_2 & z_3 & z_4 & z_5 & z_6 & z_7 & z_8 \\ z_1^2 & z_2^2 & z_3^2 & z_4^2 & z_5^2 & z_6^2 & z_7^2 & z_8^2 \\ z_1^3 & z_2^3 & z_3^3 & z_4^3 & z_5^3 & z_6^3 & z_7^3 & z_8^3 \\ z_1^4 & z_2^4 & z_3^4 & z_4^4 & z_5^4 & z_6^2 & z_7^4 & z_8^4 \\ z_1^5 & z_2^5 & z_3^5 & z_4^5 & z_5^5 & z_6^5 & z_7^5 & z_8^5 \\ z_1^6 & z_2^6 & z_3^6 & z_4^6 & z_5^6 & z_6^6 & z_7^6 & z_8^6 \\ z_1^7 & z_2^7 & z_3^7 & z_4^7 & z_5^7 & z_6^7 & z_7^7 & z_8^7 \\ z_1^8 & z_2^8 & z_3^8 & z_4^8 & z_5^8 & z_6^8 & z_7^8 & z_8^8 \end{bmatrix}$$

MAXIMUM-CONSECUTIVE-LAG ARRAYS

Corollary 4 in pre-print http://arxiv.org/abs/2606.00360

Corollary 4

If the number of sensors in the thinned array exceeds the number of sources, i.e., $M_{\mathcal{I}} > L$, then there always exists a sparse sensor configuration, obtained by removing up to $M - M_{\mathcal{I}}$ sensors from the underlying M-element ULA, whose array manifold is unambiguous, i.e., $\boldsymbol{A}_{\mathcal{I}}(z_1, \ldots, z_L) \in \mathbb{C}^{M_{\mathcal{I}} \times L}$ has full column rank for every set of distinct generators $\{z_1, \ldots, z_L\}$.

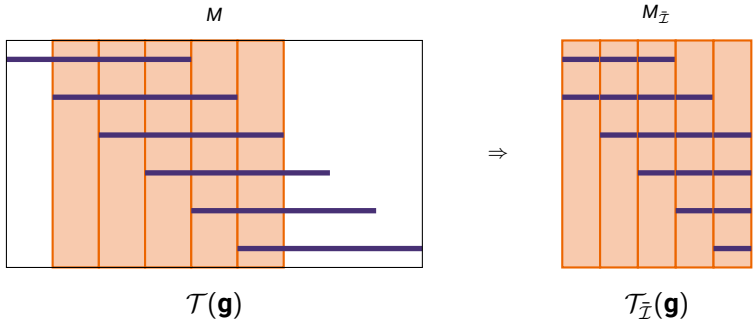
Sketch of proof

- Consider the boundary case $M_{\mathcal{I}} = L + 1$ and choose $\bar{\mathcal{I}} = \{2, 3, \ldots, M - L\}$, thus $\mathcal{I} = \{1, M - L + 1, M - L + 2, \ldots, M\}$.
- The corresponding complementary submatrix $\mathcal{T}_{\bar{\mathcal{I}}}(\boldsymbol{g}) \in \mathbb{C}^{(M-L) \times (M-L-1)}$ is upper triangular with g_0 on the main diagonal.
- Hence is has full column rank for any nonzero g_0.
- By Theorem 1, $\boldsymbol{A}_{\mathcal{I}}(z_1, \ldots, z_L) \in \mathbb{C}^{(L+1) \times L}$ has full column rank for every set of distinct generators $\{z_1, \ldots, z_L\}$.

TECHNISCHE UNIVERSITÄT DARMSTADT

MAXIMUM-CONSECUTIVE-LAG ARRAYS

Corollary 4 in pre-print http://arxiv.org/abs/2606.00360



MAXIMUM CONSECUTIVE LAG EXAMPLE

Example 9 of pre-print http://arxiv.org/abs/2606.00360

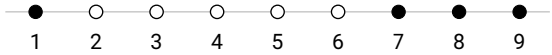
- For $M = 9$, $M_{\mathcal{I}} = 4$ and $L = 3$ sources, the Toeplitz matrix is

$$\mathcal{T}(\boldsymbol{g}) = \begin{bmatrix} g_0 & g_1 & g_1^* & g_0^* & 0 & 0 & 0 & 0 & 0 \\ 0 & g_0 & g_1 & g_1^* & g_0^* & 0 & 0 & 0 & 0 \\ 0 & 0 & g_0 & g_1 & g_1^* & g_0^* & 0 & 0 & 0 \\ 0 & 0 & 0 & g_0 & g_1 & g_1^* & g_0^* & 0 & 0 \\ 0 & 0 & 0 & 0 & g_0 & g_1 & g_1^* & g_0^* & 0 \\ 0 & 0 & 0 & 0 & 0 & g_0 & g_1 & g_1^* & g_0^* \end{bmatrix}.$$

TECHNISCHE UNIVERSITÄT DARMSTADT

MAXIMUM CONSECUTIVE LAG EXAMPLE

Example 9 of pre-print http://arxiv.org/abs/2606.00360



- The corresponding complementary submatrix is

$$\mathcal{T}_{\bar{\mathcal{I}}}(\boldsymbol{g}) = \begin{bmatrix} g_1 & g_1^* & g_0^* & 0 & 0 \\ g_0 & g_1 & g_1^* & g_0^* & 0 \\ 0 & g_0 & g_1 & g_1^* & g_0^* \\ 0 & 0 & g_0 & g_1 & g_1^* \\ 0 & 0 & 0 & g_0 & g_1 \\ 0 & 0 & 0 & 0 & g_0 \end{bmatrix}.$$

- The 5×5 submatrix formed by the last five rows of $\mathcal{T}_{\bar{\mathcal{I}}}(\boldsymbol{g})$ is upper triangular with diagonal entries $g_0 \neq 0$, so its determinant is $g_0^5 \neq 0$.
- Hence $\mathcal{T}_{\bar{\mathcal{I}}}(\boldsymbol{g})$ has full column rank for all $\boldsymbol{g}$, and by Theorem 1 the array manifold is unambiguous for $L = 3$.

Array analysis

Alternative rank criterion with fixed generators

FIXING MULTIPLE GENERATORS ON THE UNIT-CIRCLE

- We have previously discussed that any generator $z_1, \ldots, z_L$ set can always be rotated by a fixed electrical angle α, i.e., by multiplying all generators with $e^{j\alpha}$.
- This rotation property can, e.g., be used to fix one of the generators at a desired angle, i.e., to ensure for example $z_1 = 1$.
- The question that arises in this context is how multiple generators can be fixed at predefined positions on the unit-circle in the ambiguity analysis (e.g., L' generators $z_{i_1}, \ldots, z_{i_{L'}}$ out of a total of L generators).
- The following results provides an answer to this question, which turns out to be very useful in practice.

VANDERMONDE REDUCTION

Corollary 8 of pre-print http://arxiv.org/abs/2606.00360

Corollary 8

Let $\{\acute{z}_1, \ldots, \acute{z}_{L'}\} \subset \{z_1, \ldots, z_L\}$ *be an arbitrary subset of* $L' < L$ *generators of* $P_{\boldsymbol{g}}(z)$*, and let* $\boldsymbol{f}$ *be the coefficient vector of the cofactor polynomial* $P_{\boldsymbol{f}}(z)$*, so that* $P_{\boldsymbol{g}}(z) = P_{\boldsymbol{f}}(z)\, P_{\boldsymbol{h}}(z)$ *with* $P_{\boldsymbol{h}}(z) = \prod_{\ell=1}^{L'} (z - \acute{z}_\ell)$*. Then the generalized Vandermonde matrix* $\boldsymbol{A}_{\mathcal{I}}(z_1, \ldots, z_L)$ *is rank deficient if and only if the* $(M - \bar{L}') \times (M_{\bar{\mathcal{I}}} + L')$ *augmented matrix*

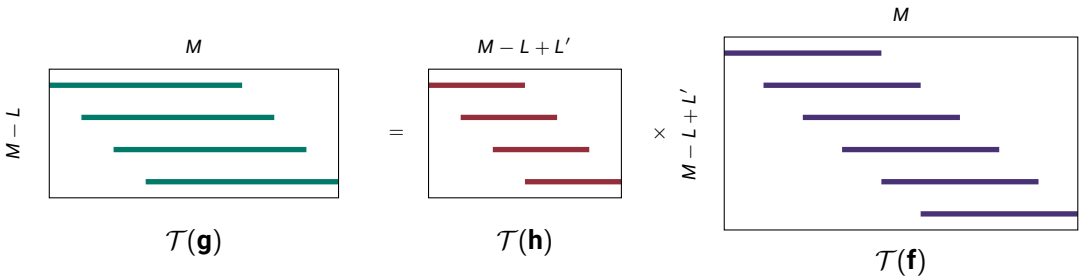
$$\mathcal{A}(\boldsymbol{f}, \acute{z}_1, \ldots, \acute{z}_{L'}) = \begin{bmatrix} \mathcal{T}_{\bar{\mathcal{I}}}(\boldsymbol{f}) & \boldsymbol{A}_{M-\bar{L}'}(\acute{z}_1, \ldots, \acute{z}_{L'}) \end{bmatrix}$$

is rank deficient.

- Corollary 8 relates rank of the generalized Vandermonde matrix $\boldsymbol{A}_{\mathcal{I}}(z_1, \ldots, z_L)$ of the thinned array and that of the full Vandermonde matrix $\boldsymbol{A}_{M-\bar{L}'}(\acute{z}_1, \ldots, \acute{z}_{L'})$ of a ULA with $M - \bar{L}'$ elements, for any arbitrary subset of L' generators.
- Characterizing ambiguities in the thinned array reduces to characterizing rank deficiencies of an augmented full-ULA steering matrix or extended complementary submatrix.

MATRIX REPRESENTATION OF POLYNOMIAL MULTIPLICATION

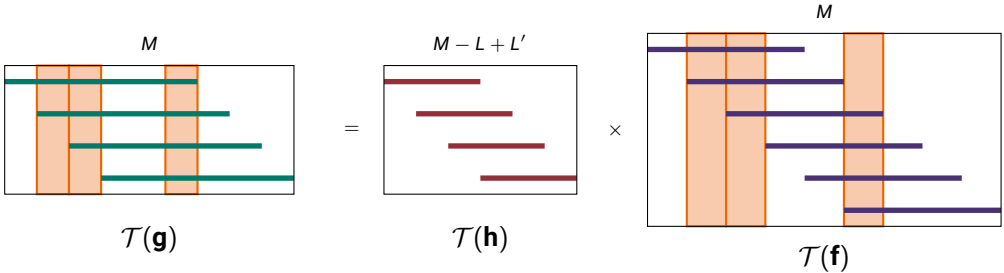
Sketch of proof of Corollary 8. For details see: http://arxiv.org/abs/2606.00360



- Prescribing L' of the L generators factors the polynomial: $P_{\mathbf{g}}(z) = P_{\mathbf{h}}(z)\, P_{\mathbf{f}}(z)$
- This lifts to the matrices: $\mathcal{T}(\mathbf{g}) = \mathcal{T}(\mathbf{h})\, \mathcal{T}(\mathbf{f})$, both spanning all M columns

LOCATE $\bar{\mathcal{I}}$ IN BOTH FACTORS

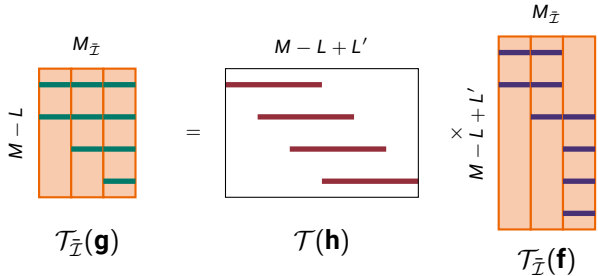
Sketch of proof of Corollary 8. For details see: http://arxiv.org/abs/2606.00360



$\mathcal{T}(\mathbf{g})$ and $\mathcal{T}(\mathbf{f})$ share the same M columns and $\bar{\mathcal{I}}$ removal positions

EXTRACTING THOSE COLUMNS

Sketch of proof of Corollary 8. For details see: http://arxiv.org/abs/2606.00360

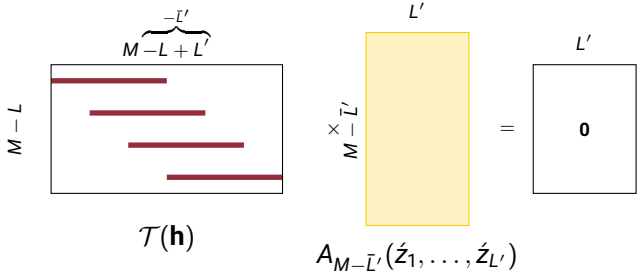


$\mathcal{T}(\mathbf{g})$ and $\mathcal{T}(\mathbf{f})$ share the same M columns and $\bar{\mathcal{I}}$ removal positions

TECHNISCHE UNIVERSITÄT DARMSTADT

THE NULL SPACE OF $\mathcal{T}(\mathbf{h})$

Sketch of proof of Corollary 8. For details see: http://arxiv.org/abs/2606.00360



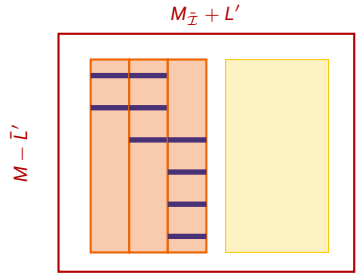
Because $\mathcal{T}(\mathbf{h})$ is built from the L' prescribed generators, its null space is exactly the Vandermonde matrix of those generators, with $M-\bar{L}'=M-L+L'$ rows matching $\mathcal{T}(\mathbf{h})$'s column count

TECHNISCHE UNIVERSITÄT DARMSTADT

THE AUGMENTED MATRIX

Sketch of proof of Corollary 8. For details see: http://arxiv.org/abs/2606.00360

$$\mathcal{A}(\mathbf{f}, \acute{z}_1, \ldots, \acute{z}_{L'})$$



$$\mathcal{T}_{\bar{\mathcal{I}}}(\boldsymbol{f}) \cap \mathcal{R}\big(\boldsymbol{A}_{M-\bar{L}'}(\acute{z}_1, \ldots, \acute{z}_{L'})\big) \neq \emptyset$$

$$\Updownarrow$$

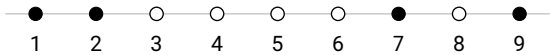
$\mathcal{A}(\mathbf{f}, \acute{z}_1, \ldots, \acute{z}_{L'})$ **rank deficient**

$$\Updownarrow$$

$\mathbf{A}_{\mathcal{I}}(z_1, \ldots, z_L)$ **is rank deficient**

SPARSE ARRAY WITH LOW MUTUAL COUPLING

Example 10 of pre-print http://arxiv.org/abs/2606.00360



- As an alternative to Example 9, we seek a sparse design with $M_{\mathcal{I}} = 4$ sensors and $L = 3$ sources that avoids the dense contiguous lag three-element subarray.
- For this array the complementary submatrix is

$$\mathcal{T}_{\bar{\mathcal{I}}}(\boldsymbol{g}) = \begin{bmatrix} g_1^* & g_0^* & 0 & 0 & 0 \\ g_1 & g_1^* & g_0^* & 0 & 0 \\ g_0 & g_1 & g_1^* & g_0^* & 0 \\ 0 & g_0 & g_1 & g_1^* & 0 \\ 0 & 0 & g_0 & g_1 & g_0^* \\ 0 & 0 & 0 & g_0 & g_1^* \end{bmatrix}.$$

- Rather than directly analyzing the rank of $\mathcal{T}_{\bar{\mathcal{I}}}(\boldsymbol{g})$, we apply Corollary 8 with one prescribed generator $\acute{z}_1 = 1$, i.e $L' = 1$.
- The cofactor polynomial has degree $L - L' = 2$ with coefficient vector $\boldsymbol{f} = [f_0, f_1, f_0^*]^{\mathsf{T}}$.

SPARSE ARRAY WITH LOW MUTUAL COUPLING

Example 10 of pre-print http://arxiv.org/abs/2606.00360

- The associated 7×6 augmented complementary submatrix reads

$$\mathcal{A}(\boldsymbol{f}, \acute{z}_1) = \begin{bmatrix} f_0^* & 0 & 0 & 0 & 0 & 1 \\ f_1 & f_0^* & 0 & 0 & 0 & 1 \\ f_0 & f_1 & f_0^* & 0 & 0 & 1 \\ 0 & f_0 & f_1 & f_0^* & 0 & 1 \\ 0 & 0 & f_0 & f_1 & 0 & 1 \\ 0 & 0 & 0 & f_0 & f_0^* & 1 \\ 0 & 0 & 0 & 0 & f_1 & 1 \end{bmatrix}.$$

- It can be shown that the augmented matrix $\mathcal{A}(\boldsymbol{f}, \acute{z}_1)$ has full column rank, i.e., $\operatorname{rank}\big(\mathcal{A}(\boldsymbol{f}, \acute{z}_1)\big) = 6$ for all $|f_0| = 1$ and $f_1 \in \mathbb{R}$.
- It follows that the associated generalized Vandermonde matrix $\boldsymbol{A}_{\mathcal{I}}(z_1, z_2, z_3)$ is always full rank.

SETUP: LINEAR INDEPENDENCE OF THE FIRST FIVE COLUMNS

Example 10 of pre-print http://arxiv.org/abs/2606.00360

$$\mathcal{A}(\boldsymbol{f}, \acute{z}_1) = \begin{bmatrix} f_0^* & 0 & 0 & 0 & 0 & 1 \\ f_1 & f_0^* & 0 & 0 & 0 & 1 \\ f_0 & f_1 & f_0^* & 0 & 0 & 1 \\ 0 & f_0 & f_1 & f_0^* & 0 & 1 \\ 0 & 0 & f_0 & f_1 & 0 & 1 \\ 0 & 0 & 0 & f_0 & f_0^* & 1 \\ 0 & 0 & 0 & 0 & f_1 & 1 \end{bmatrix}.$$

Since $f_0 = e^{\mathrm{j}\phi_0} \neq 0$, the first five columns of $\mathcal{A}(\boldsymbol{f}, \acute{z}_1)$ are linearly independent:

- the first column has f_0^* in row 1 as a pivot;
- the others can be reduced to staircase form by elementary operations using f_0 on the subdiagonals.

Hence rank deficiency can occur only if the sixth column (the all-ones Vandermonde column for $\acute{z}_1 = 1$) lies in the span of the first five.

SETTING UP THE LINEAR SYSTEM

Example 10 of pre-print http://arxiv.org/abs/2606.00360

Suppose, for some linear coefficients $\alpha_1, \ldots, \alpha_5 \in \mathbb{C}$,

$$\sum_{j=1}^{5} \alpha_j \, [\mathcal{A}(\boldsymbol{f}, \acute{z}_1)]_{:,j} = [\mathcal{A}(\boldsymbol{f}, \acute{z}_1)]_{:,6} = \mathbf{1}.$$

Writing out the seven row equations gives:

$$\begin{aligned}
f_0^* \, \alpha_1 &= 1, \\
f_1 \, \alpha_1 + f_0^* \, \alpha_2 &= 1, \\
f_0 \, \alpha_1 + f_1 \, \alpha_2 + f_0^* \, \alpha_3 &= 1, \\
f_0 \, \alpha_2 + f_1 \, \alpha_3 + f_0^* \, \alpha_4 &= 1, \\
f_0 \, \alpha_3 + f_1 \, \alpha_4 &= 1, \\
f_0 \, \alpha_4 + f_0^* \, \alpha_5 &= 1, \\
f_1 \, \alpha_5 &= 1.
\end{aligned}$$

The first, second, third, fourth, and sixth equations can be solved *recursively* for $\alpha_1, \ldots, \alpha_5$ in terms of f_0 and f_1. Substituting into the two remaining equations (the fifth and seventh) yields two consistency conditions.

CONSISTENCY CONDITIONS

Example 10 of pre-print http://arxiv.org/abs/2606.00360

Substituting the recursively-solved α_j into the remaining two equations yields

$$F_5(f_0, f_1) = 0, \qquad F_7(f_0, f_1) = 0,$$

where

$$\begin{aligned} F_5(f_0, f_1) &= -f_0^4 f_1^4 + f_0^3 f_1^3 + (3f_0^4 - f_0^2) f_1^2 \\ &\quad + (-2f_0^3 + f_0) f_1 + (f_0^2 - f_0^4 - 1), \\ F_7(f_0, f_1) &= f_0^6 f_1^4 - f_0^5 f_1^3 + (-2f_0^6 + f_0^4) f_1^2 \\ &\quad + (f_0^5 - f_0^3 + f_0) f_1 - 1. \end{aligned}$$

Goal: show these have no common solution with $|f_0| = 1$, $f_1 \in \mathbb{R}$ (proof by contradiction).

PROOF BY CONTRADICTION

Example 10 of pre-print http://arxiv.org/abs/2606.00360

Define

$$F(f_0, f_1) = F_7(f_0, f_1) + f_0^2 F_5(f_0, f_1).$$

The f_1^4 and f_1^3 terms cancel, and direct simplification gives

$$F(f_0, f_1) = f_0^6 f_1^2 - f_0^6 - f_0^5 f_1 + f_0^4 - f_0^2 + f_0 f_1 - 1.$$

TRIGONOMETRIC FORM

Example 10 of pre-print http://arxiv.org/abs/2606.00360

Writing $f_0 = e^{\mathrm{j}\phi_0}$ and dividing $F(f_0, f_1) = 0$ by $f_0^3 = e^{\mathrm{j}3\phi_0}$:

$$\begin{aligned}(f_1^2 - 1)\, e^{\mathrm{j}3\phi_0} - f_1\, e^{\mathrm{j}2\phi_0} + e^{\mathrm{j}\phi_0} - e^{-\mathrm{j}\phi_0} \\ + f_1\, e^{-\mathrm{j}2\phi_0} - e^{-\mathrm{j}3\phi_0} = 0.\end{aligned}$$

Separating real and imaginary parts:

$$(f_1^2 - 2)\, \cos(3\phi_0) = 0, \tag{1}$$

$$f_1^2\, \sin(3\phi_0) - 2f_1\, \sin(2\phi_0) + 2\, \sin(\phi_0) = 0. \tag{2}$$

CASE 1: $\cos(3\phi_0) \neq 0$

Example 10 of pre-print http://arxiv.org/abs/2606.00360

Equation (1) forces $f_1^2 = 2$. Substituting into (2) gives

$$\sin(3\phi_0) - f_1 \sin(2\phi_0) + \sin(\phi_0) = 0.$$

Using $\sin(3\phi_0) + \sin(\phi_0) = 2\sin(2\phi_0)\cos(\phi_0)$, this reduces to

$$\sin(2\phi_0)\left(2\cos(\phi_0) - f_1\right) = 0,$$

giving two sub-cases.

(a) $\sin(2\phi_0) = 0$

$\phi_0 \in \{0, \pi\}$ ($\pm\pi/2$ excluded), so $f_0 = \pm 1$, $f_1 = \pm\sqrt{2}$.

$$F_5(\pm 1, \pm\sqrt{2}) \neq 0.$$

(b) $f_1 = 2\cos(\phi_0)$

With $f_1^2 = 2$: $\cos(\phi_0) = \pm\frac{1}{\sqrt{2}}$, so $\phi_0 \in \{\pm\pi/4, \pm 3\pi/4\}$.

$$F_5(f_0, f_1) \neq 0.$$

Both sub-cases contradict the required consistency conditions.

CASE 2: $\cos(3\phi_0) = 0$

Example 10 of pre-print http://arxiv.org/abs/2606.00360

Then

$$\phi_0 \in \left\{ \tfrac{\pi}{6},\ \tfrac{\pi}{2},\ \tfrac{5\pi}{6},\ \tfrac{7\pi}{6},\ \tfrac{3\pi}{2},\ \tfrac{11\pi}{6} \right\}.$$

Substituting each value into (2) yields either:

- the quadratic $f_1^2 \pm \sqrt{3} f_1 + 1 = 0$, which has *no real solution* (discriminant $3 - 4 = -1 < 0$), or
- $f_0 = \pm \mathrm{j}$ together with $f_1^2 = 2$.

In the latter case, direct substitution shows $F_5(\pm \mathrm{j}, \pm\sqrt{2}) \neq 0$, again a contradiction.

CONCLUSION

In every case, assuming $F_5(f_0, f_1) = F_7(f_0, f_1) = 0$ with $|f_0| = 1$, $f_1 \in \mathbb{R}$ leads to a contradiction.

Thus $F_5(f_0, f_1)$ and $F_7(f_0, f_1)$ cannot vanish simultaneously for any $|f_0| = 1$ and $f_1 \in \mathbb{R}$.

Conclusion: the sixth column of $\mathcal{A}(\boldsymbol{f}, \acute{z}_1)$ never lies in the span of the first five columns, and

$$\text{rank}\left(\mathcal{A}(\boldsymbol{f}, \acute{z}_1)\right) = 6$$

for all admissible f_0 and f_1.

Sparse array design

Coarrays

TECHNISCHE UNIVERSITÄT DARMSTADT

SPARSE ARRAYS: COVARIANCE & RECOVERY

- Signal model: $\boldsymbol{y} = \boldsymbol{A}(z_1, \ldots, z_L)\,\boldsymbol{x} \quad +$ noise
- Observation: Covariance matrix holds lags that are "invisible" to the steering vector!

$$\mathbf{R}_{\text{ULA}} = \begin{bmatrix} 1 & z^{-1} & z^{-2} & z^{-3} & z^{-4} \\ z^{1} & 1 & z^{-1} & z^{-2} & z^{-3} \\ z^{2} & z^{1} & 1 & z^{-1} & z^{-2} \\ z^{3} & z^{2} & z^{1} & 1 & z^{-1} \\ z^{4} & z^{3} & z^{2} & z^{1} & 1 \end{bmatrix}$$

thinned →

$$\mathbf{R}_{\text{sparse}} = \begin{bmatrix} 1 & z^{-1} & z^{-2} & & z^{-4} \\ z^{1} & 1 & z^{-1} & & z^{-3} \\ z^{2} & z^{1} & 1 & & z^{-2} \\ & & & & \\ z^{4} & z^{3} & z^{2} & & 1 \end{bmatrix}$$

- $\mathbf{R} = \mathbb{E}\left\{\mathbf{y}\mathbf{y}^{\mathrm{H}}\right\} = \mathbf{a} R_{xx} \mathbf{a}^{\mathrm{H}}$ is the covariance matrix of the noiseless measurements
- $\mathbf{R} = \mathbf{a}\mathbf{a}^{\mathrm{H}}$ shown here for a single source

SPARSE ARRAYS: DIFFERENCE CO-ARRAYS (DCA)

- Extracting the entries via an aggregation matrix $\mathbf{J}$ defines the Difference Co-Array (DCA)

$$\mathbf{a}_{\text{Co-Array}} = \mathbf{J}\,\text{vec}\left(\mathbf{R}_{\text{sparse}}\right)$$

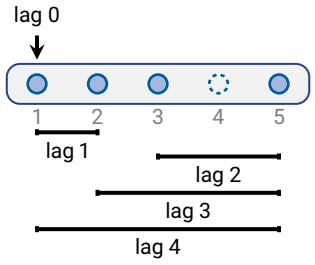
Physical Array		**Difference Co-Array**	
Aperture:	D	Aperture:	$2D-1$
Sensors:	M	Sensors:	$M_{\text{Co-Array}} > M$

- Larger virtual aperture
- More virtual sensors

$\Longrightarrow$ Increased number of resolvable sources

$\Longrightarrow$ Question: How many sources can be resolved unambiguously?

SPARSE ARRAYS: DIFFERENCE CO-ARRAYS (DCA)



List of unique lags:

$$\{-4,\ -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3,\ 4\}$$

$$\mathbf{R}_{\text{sparse}}$$

$$\begin{bmatrix} 1 & z^{-1} & z^{-2} & & z^{-4} \\ z^{1} & 1 & z^{-1} & & z^{-3} \\ z^{2} & z^{1} & 1 & & z^{-2} \\ & & & & \\ z^{4} & z^{3} & z^{2} & & 1 \end{bmatrix}$$

$$\mathbf{a}_{\text{Co-Array}}$$

$$\begin{bmatrix} z^{-4} & z^{-3} & z^{-2} & z^{-1} & 1 & z^{1} & z^{-2} & z^{3} & z^{4} \end{bmatrix}^{\mathrm{T}}$$

SPARSE ARRAYS: CO-ARRAY SIGNAL MODEL

- Suppose L uncorrelated sources such that $\mathbf{R_{xx}} = \mathrm{diag}(\mathbf{p}),\ \mathbf{p} = [P_1\ P_2\ \ldots\ P_L]^{\mathsf{T}}$
- Including ZMCSCG noise with covariance $\mathbf{R_{nn}} = \sigma_n^2\mathbf{I}$, the overall covariance can be written as

$$\mathbf{R_{xx}} = \mathbf{A}\mathbf{R_{xx}}\mathbf{A}^{\mathsf{H}} + \sigma_n^2\mathbf{I} = \sum_{\ell=1}^{L} P_\ell \mathbf{a}(z_\ell)\mathbf{a}^{\mathsf{H}}(z_\ell) + \sigma_n^2\mathbf{I}$$

- Applying the $\mathrm{vec}(\mathbf{R_{xx}})$ operator yields and the selection operator $\mathbf{J}$ [14]

$$\begin{aligned}\mathbf{J}\,\mathrm{vec}(\mathbf{R_{xx}}) &= \sum_{\ell=1}^{L} P_\ell \mathbf{J}\,(\mathbf{a}^*(z_\ell) \otimes \mathbf{a}(z_\ell)) + \sigma_n^2 \mathbf{J}\,\mathrm{vec}(\mathbf{I}) \\ &= \mathbf{A}_{\text{Co-Array}}\mathbf{p} + \sigma_n^2 \mathbf{J}\,\mathrm{vec}(\mathbf{I})\end{aligned}$$

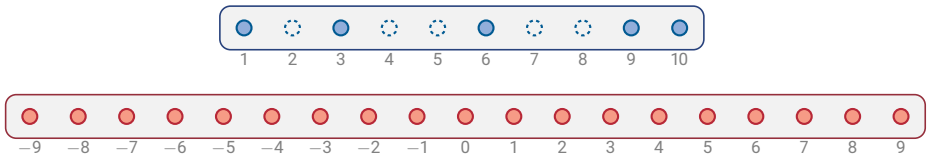
- $\mathrm{vec}(\mathbf{R_{xx}})$ behaves as a received signal vector with virtual array steering matrix $\mathbf{A}_{\text{Co-Array}}$
- Source signals are replaced by their powers and noise is deterministic

[14] M. G. Amin, "Sparse Arrays for Radar, Sonar, and Communications", 2024

TECHNISCHE UNIVERSITÄT DARMSTADT

SPARSE ARRAYS: MINIMUM REDUNDANCY ARRAY (MRA)

- **Idea:** Minimize the number of physical sensors (i.e., redundant lags) required to obtain a contiguous co-array of a given aperture[15][3]
- **Co-Array:** Hole-free DCA: Every lag up to the maximum is present
- **Construction:** No closed-form solution, sensor positions found by combinatorial search while preserving the capability to generate all lags
- **Drawback:** Computationally expensive to find, does not scale well to large M



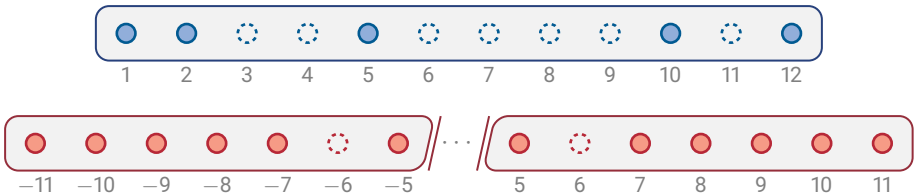
[15] A. Moffet, "Minimum-redundancy linear arrays", 1968

[3] P. Pal and P. P. Vaidyanathan, "Nested Arrays: A Novel Approach to Array Processing With Enhanced Degrees of Freedom", 2010

TECHNISCHE UNIVERSITÄT DARMSTADT

SPARSE ARRAYS: MINIMUM HOLE ARRAY (MHA) / GOLOMB RULER

- **Idea:** For a fixed number of sensors M, minimize holes in the difference co-array[16][17]
- **Co-Array:** Few isolated missing lags, but every lag only appears once
- **Construction:** Also search-based, no general closed-form expression
- **Relation to MRA:** Maximum contiguous lag (MRA) vs. minimal occurrence of lags (MHA)
- Combination of MRA and MHA is called perfect, but only exists for $M \leq 4$

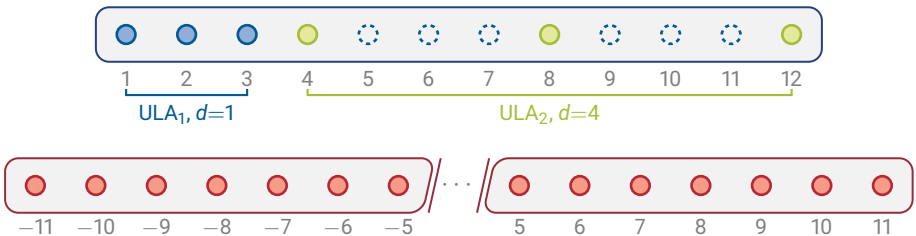


[16] G. Bloom and S. Golomb, "Applications of numbered undirected graphs", 1977

[17] X. Li et al., "Sparse Linear Arrays for Direction-of-Arrival Estimation: A Tutorial Overview", 2025

SPARSE ARRAYS: NESTED ARRAY (NA)

- **Idea:** Concatenate two ULA sub-arrays: Dense inner ULA of M_1 sensors with spacing d and sparse outer ULA of M_2 sensors with spacing $(M_1+1)d$[3]
- **Co-Array:** Contiguous DCA with $O(M_1M_2)$ virtual sensors
- **Construction:** Closed-form: sensor positions known analytically
- **Drawback:** Some lag redundancy, mutual coupling between dense inner sensors

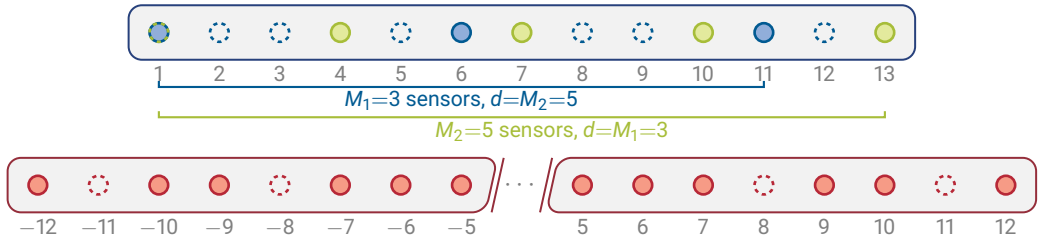


[3] P. Pal and P. P. Vaidyanathan, "Nested Arrays: A Novel Approach to Array Processing With Enhanced Degrees of Freedom", 2010

TECHNISCHE UNIVERSITÄT DARMSTADT

SPARSE ARRAYS: COPRIME ARRAY (CA)

- **Idea:** Two ULA sub-arrays with coprime inter-sensor spacings M_1 and M_2 ($\gcd(M_1, M_2) = 1$), with M_1 and M_2 sensors respectively[4]
- **Co-Array:** $O(M_1 M_2)$ virtual sensors, large contiguous region
- **Construction:** Closed-form: sensor positions given directly by M_1, M_2
- **Advantage:** Reduced mutual coupling compared to NA; flexible design via choice of M_1, M_2



[4] P. P. Vaidyanathan and P. Pal, "Sparse Sensing With Co-Prime Samplers and Arrays", 2011

TECHNISCHE UNIVERSITÄT DARMSTADT

IMPORTANT CRITERIA FOR SPARSE ARRAY DESIGN

- Optimized beampattern[17]
  - Control mainlobe width and / or maximize virtual aperture
- Maximized degrees of freedom (DOF)[3]
  - DOF determined by the difference co-array (DCA)
  - Number of resolvable sources $\leq$ effective DCA aperture / virtual elements
- Mitigation of mutual coupling[18]
  - Distorts array manifold and degrades estimation accuracy
- Optimized estimation performance bounds[1], e.g., Cramér–Rao Bound (CRB)[2]
  - Directly linked to achievable parameter estimation accuracy

[17] X. Li et al., "Sparse Linear Arrays for Direction-of-Arrival Estimation: A Tutorial Overview", 2025
[3] P. Pal and P. P. Vaidyanathan, "Nested Arrays: A Novel Approach to Array Processing With Enhanced Degrees of Freedom", 2010
[18] C. Larmour et al., "Sparse Array Mutual Coupling Reduction", 2024
[1] J. Tabrikian et al., "Cognitive Antenna Selection for Automotive Radar Using Bobrovsky-Zakai Bound", 2021
[2] I. van der Werf et al., "Jointly Optimal Array Geometries and Waveforms in Active Sensing: New Insights Into Array Design via the Cramér-Rao Bound", 2025

Sparse array design

Ambiguity-based array design

RETURNING TO THEOREM 1

Insights on sparse array design

An immediate question arises:

Can the rank-drop property of Theorem 1 be used to gain insights into design of ambiguity-free sparse arrays?

The answer is affirmative:

- Considering specific choices of coefficient vectors $\boldsymbol{g}$, entire columns can become zero, or a linear dependence between specific columns of the complementary submatrix $\mathcal{T}_{\bar{\mathcal{I}}}(\boldsymbol{g})$ can be created.
- The sensors corresponding to zero or redundant columns cannot jointly be deleted.
- The design rules obtained from this consideration largely reduce the search-space of possible ambiguity-free sparse array configurations.

TECHNISCHE UNIVERSITÄT DARMSTADT

COROLLARY 5: FORBIDDEN CENTER ELEMENTS

$2L \geq M+1$: *removing* ***any*** *element of* $Q = \{M-L+1, \ldots, L\}$ *causes ambiguity.*

$$M = 11, \;\; L = 8$$

Affected submatrix (all of $Q = \{4, \ldots, 8\}$ removed, $\bar{\mathcal{I}} = Q$):

$$\mathcal{T}_{\bar{\mathcal{I}}}(\mathbf{g}) = \begin{bmatrix} g_3 & g_4 & g_3^* & g_2^* & g_1^* \\ g_2 & g_3 & g_4 & g_3^* & g_2^* \\ g_1 & g_2 & g_3 & g_4 & g_3^* \end{bmatrix}$$

Every column of $\mathcal{T}_{\bar{\mathcal{I}}}(\mathbf{g})$ is built entirely from $g_1, \ldots, g_4$.

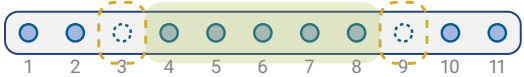
Rank deficient: choose, e.g.,

$$g_0 = g_0^* = 1, \; g_k = 0 \; \forall k \neq 0$$

Then $\mathcal{T}_{\bar{\mathcal{I}}}(\tilde{\mathbf{g}}) = \mathbf{0}_{3\times 5}$, the *entire* submatrix vanishes, so $\mathrm{rank} = 0$ (maximal deficiency).

COROLLARY 6: MUTUALLY EXCLUSIVE INDICES (TYPE A)

$2L \geq M - 1$: *removing* ***both*** $M-L$ *and* $L+1$ *simultaneously causes ambiguity.*



$M = 11, \;\; L = 8$

Shaded region (4–8): already forced retained by Corollary 5

Affected submatrix ($\bar{\mathcal{I}} = \{3, 9\}$):

$$\mathcal{T}_{\bar{\mathcal{I}}}(\mathbf{g}) = \begin{bmatrix} g_2 & g_0^* \\ g_1 & g_1^* \\ g_0 & g_2^* \end{bmatrix}$$

Columns 3 and 9 share the same coefficients in mirrored order.

Rank deficient: choose, e.g,

$$\mathbf{g} = \mathbf{1} \;\; (g_0 = \cdots = g_4 = 1)$$

Columns 3, 9 both equal $[1, 1, 1]^T$.

COROLLARY 7: MUTUALLY EXCLUSIVE INDICES (TYPE B)

$$N = M - L, \qquad 0 \le p \le K, \qquad p+1 \le q \le L-p, \qquad N \le L + q - 2p - 1$$

*Removing **both** $S_{p,q} = \{q,\ L-p+q\}$ simultaneously causes ambiguity.*

$$M = 11, \ \ L = 8$$

Choosing $p = 1,\ q = 2$ returns $\bar{\mathcal{I}} = \{2, 9\}$:

$$\mathcal{T}_{\bar{\mathcal{I}}}(\mathbf{g}) = \begin{bmatrix} g_1 & g_0^* \\ g_0 & g_1^* \\ 0 & g_2^* \end{bmatrix}$$

Rank deficient: choose, e.g.,

$$g_0 = g_1 = 1, \ \ g_2 = g_4 = 0 \ \ g_3 \text{ free}$$

Columns 2, 9 both equal $[1, 1, 0]^T$.

Conclusions and outlook

CONCLUSIONS AND OUTLOOK

We addressed a fundamental problem:

- The Kruskal rank (Spark) of Generalized Vandermonde matrix
- A scalable and complete characterization of rank drop conditions
- Results can be used for sparse array analysis and design

Future work:

- Optimal sparse array design: Find further results
- Sparse co-array design
- Sparse 2D-array analysis and design
- More general arrays: Shift invariant arrays, partly calibrated arrays
- Consider particular array cases
- Transfer results to other applications: System identification in control,...

Please check the associated pre-print: http://arxiv.org/abs/2606.00360

TECHNISCHE UNIVERSITÄT DARMSTADT

ACKNOWLEDGEMENTS

This work was supported by the German Federal Ministry of Research, Technology and Space through Project “Open6GHub+” under Grant16KIS2.

I would like to give special thanks to my doctoral student Paul Armand Ugo Lysandre Göttlich (paul.goettlich@nt.tu-darmstadt.de) for his help and the many fruitful discussions in the preparation of these slides.

TECHNISCHE UNIVERSITÄT DARMSTADT

ANY QUESTIONS ?

Find more details in the pre-print: http://arxiv.org/abs/2606.00360

Please contact us if you have ideas and interest for collaborations!

Contact: Marius Pesavento – pesavento@ieee.org

REFERENCES I

[1] J. Tabrikian, O. Isaacs, and I. Bilik, "Cognitive antenna selection for automotive radar using Bobrovsky-Zakai bound", *IEEE J. Sel. Top. Signal Process.*, vol. 15, no. 4, pp. 892–903, 2021, DOI: 10.1109/JSTSP.2021.3071586.

[2] I. van der Werf, G. Leus, and R. Rajamäki, "Jointly optimal array geometries and waveforms in active sensing: New insights into array design via the cramér-rao bound", in *Proc. IEEE ICASSP'25*, 2025, pp. 1–5. DOI: 10.1109/ICASSP49660.2025.10890202.

[3] P. Pal and P. P. Vaidyanathan, "Nested arrays: A novel approach to array processing with enhanced degrees of freedom", *IEEE Transactions on Signal Processing*, vol. 58, no. 8, pp. 4167–4181, 2010, DOI: 10.1109/TSP.2010.2049264.

[4] P. P. Vaidyanathan and P. Pal, "Sparse sensing with co-prime samplers and arrays", *IEEE TSP*, vol. 59, no. 2, pp. 573–586, 2011, DOI: 10.1109/TSP.2010.2089682.

[5] X. Wang, M. Amin, and X. Cao, "Analysis and design of optimum sparse array configurations for adaptive beamforming", *IEEE Transactions on Signal Processing*, vol. 66, no. 2, pp. 340–351, 2018, DOI: 10.1109/TSP.2017.2760279.

REFERENCES II

[6] M. Wax and I. Ziskind, "On unique localization of multiple sources by passive sensor arrays", *Proc. IEEE TASSP*, vol. 37, no. 7, pp. 996–1000, Jul. 1989, DOI: 10.1109/29.32277.

[7] A. Manikas and C. Proukakis, "Modeling and estimation of ambiguities in linear arrays", *IEEE Transactions on Signal Processing*, vol. 46, no. 8, pp. 2166–2179, Aug. 1998, DOI: 10.1109/78.705428.

[8] Y. Abramovich, N. Spencer, and A. Gorokhov, "Identifiability and manifold ambiguity in DOA estimation for nonuniform linear antenna arrays", in *Proc. IEEE ICASSP'99*, vol. 5, Mar. 1999, 2845–2848 vol.5. DOI: 10.1109/ICASSP.1999.761355.

[9] B. Friedlander, "Multi-signal ambiguity in direction-of-arrival estimation", in *2023 57th Asilomar Conference on Signals, Systems, and Computers*, 2023, pp. 1259–1263. DOI: 10.1109/IEEECONF59524.2023.10477080.

[10] N. Dowlut, "An extended ambiguity criterion for array design", in *Sensor Array and Multichannel Signal Processing Workshop Proceedings, 2002*, 2002, pp. 189–193. DOI: 10.1109/SAM.2002.1191026.

[11] F. Matter, T. Fischer, M. Pesavento, and M. E. Pfetsch, "Ambiguities in doa estimation with linear arrays", *IEEE TSP*, vol. 70, pp. 4395–4407, 2022, DOI: 10.1109/TSP.2022.3200548.

REFERENCES III

[12] S. De Marchi, "Polynomials arising in factoring generalized Vandermonde determinants: An algorithm for computing their coefficients", *Mathematical and Computer Modelling*, vol. 34, no. 3–4, pp. 271–281, 2001. DOI: 10.1016/S0895-7177(01)00060-7.

[13] M. Marden, "Geometry of Polynomials" (Mathematical Surveys). American Mathematical Soc., 1966, DOI: 10.1090/surv/003.

[14] M. G. Amin, "Sparse Arrays for Radar, Sonar, and Communications", 1st. Hoboken, NJ, USA: Wiley-IEEE Press, 2024, DOI: 10.1002/9781394191048.

[15] A. Moffet, "Minimum-redundancy linear arrays", *IEEE TAP*, vol. 16, no. 2, pp. 172–175, 1968, DOI: 10.1109/TAP.1968.1139138.

[16] G. Bloom and S. Golomb, "Applications of numbered undirected graphs", *Proceedings of the IEEE*, vol. 65, no. 4, pp. 562–570, 1977. DOI: 10.1109/PROC.1977.10517.

[17] X. Li et al., "Sparse linear arrays for direction-of-arrival estimation: A tutorial overview", *IEEE Aerospace and Electronic Systems Magazine*, pp. 1–25, 2025, DOI: 10.1109/MAES.2025.3527917.

REFERENCES IV

[18] C. Larmour, N. Buchanan, V. Fusco, and M. Ali Babar Abbasi, "Sparse array mutual coupling reduction", *IEEE Open Journal of Antennas and Propagation*, vol. 5, no. 1, pp. 201–216, 2024. DOI: 10.1109/OJAP.2023.3339368.

[19] A. Manikas, "Differential Geometry in Array Processing". London: Imperial College Press, 2004, DOI: https://dl.acm.org/doi/10.5555/1096871.1096873.

[20] Y. Abramovich, V. Gaitsgory, and N. Spencer, "Stability of manifold ambiguity resolution in DOA estimation with nonuniform linear antenna arrays", in *Proc. IEEE ICASSP'00*, vol. 5, Jun. 2000, pp. 3117–3120. DOI: 10.1109/ICASSP.2000.861197.

[21] C.-L. Liu and P. P. Vaidyanathan, "Super nested arrays: Linear sparse arrays with reduced mutual coupling—part i: Fundamentals", *IEEE Transactions on Signal Processing*, vol. 64, no. 15, pp. 3997–4012, 2016, DOI: 10.1109/TSP.2016.2558159.

[22] C.-L. Liu and P. P. Vaidyanathan, "Super nested arrays: Linear sparse arrays with reduced mutual coupling—part ii: High-order extensions", *IEEE Transactions on Signal Processing*, vol. 64, no. 16, pp. 4203–4217, 2016, DOI: 10.1109/TSP.2016.2558167.

TECHNISCHE UNIVERSITÄT DARMSTADT

REFERENCES V

[23] M. Pesavento, M. Trinh-Hoang, and M. Viberg, "Three More Decades in Array Signal Processing Research: An optimization and structure exploitation perspective", *IEEE SPM*, vol. 40, no. 4, pp. 92–106, Jun. 2023, DOI: 10.1109/MSP.2023.3255558.