Analysis vs Synthesis - An Investigation of (Co)sparse Signal Models on Graphs

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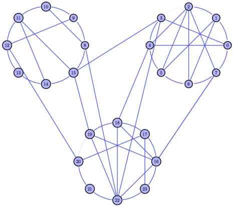
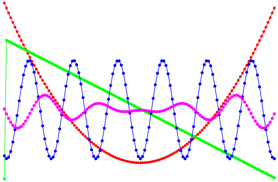
Motivation and Objectives





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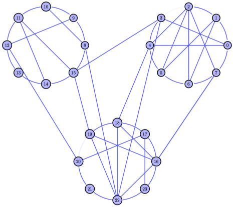

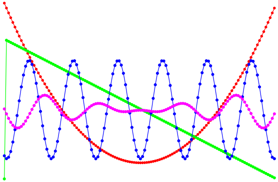

- Characterize **Sparsity on Graphs** - w.r.t. the graph connectivity & defining subspaces
- Signal Models: Tackle **Analysis vs Synthesis Problem** in the structured setting of graphs

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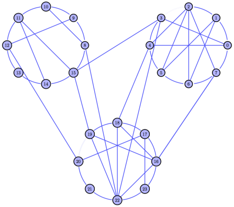

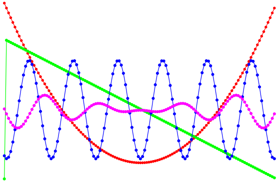

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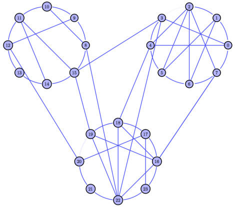

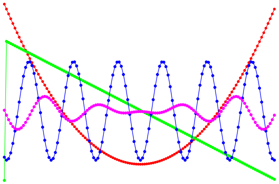

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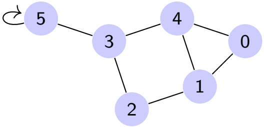
For circulant graphs

- Develop closed-form expressions of functions defining the subspaces & concretize discrepancy
- Transition between model equivalence and non-equivalence for the parametric graph Laplacian
- Unify results to quantify uniqueness guarantees for signals in UoS models on graphs

⇒ Links between Graph Theory, PDEs & Linear Algebra render problem investigation feasible

Preliminaries: Signal Processing on Graphs

- A graph $G = (V, E)$ is defined by a vertex set $V = \{0, ..., N-1\}$, with $|V| = N$, and edge set $E = \{E_0, ..., E_{M-1}\}$



- The adjacency matrix $\mathbf{A}$ captures the connectivity of G, with

$$A_{i,j} > 0, \text{ if } i \text{ and } j \text{ are adjacent } (i \neq j), \quad A_{i,j} = 0, \text{ otherwise}$$

and $\mathbf{D}$ is the diagonal degree matrix with $D_{i,i} = \sum_j A_{i,j}$

- The non-normalized graph Laplacian is given by $\mathbf{L} = \mathbf{D} - \mathbf{A}$
- The oriented incidence matrix $\mathbf{S} \in \mathbb{R}^{|E| \times |V|}$ has entries

$$S_{k,i} = \sqrt{A_{i,j}}, \; S_{k,j} = -\sqrt{A_{i,j}}, \text{ if edge } E_k = \{i, j\} \text{ is directed as } i \rightarrow j$$

and we have $\mathbf{L} = \mathbf{S}^T \mathbf{S}$

- We consider undirected, and (un-)weighted graphs without self-loops
- The graph signal $\mathbf{x}$ on G, with $\mathbf{x} : V \rightarrow \mathbb{C}$ s.t. $x(i)$ is the sample value of $\mathbf{x} \in \mathbb{C}^N$ at vertex i, is **piecewise smooth** w.r.t. $\mathbf{L}$ if $\mathbf{Lx}$ is sparse, i.e. $||\mathbf{Lx}||_0 \ll N$

The Analysis vs Synthesis Problem

Synthesis

- generate signal $\mathbf{x} = \mathbf{Dc}$, given dictionary $\mathbf{D} \in \mathbb{R}^{N \times M}$, $N \leq M$, and $\mathbf{c} \in \mathbb{R}^M$ with $||\mathbf{c}||_0 = k \ll M$ of sparse support Λ^c
  - subspace: $V_{\Lambda^c} := span(\mathbf{D}_j,\ j \in \Lambda^c)$

Analysis

- given analysis operator $\mathbf{\Omega} \in \mathbb{R}^{M \times N}$, apply constraint $||\mathbf{\Omega x}||_0 = k \ll M$ with $\mathbf{\Omega}_\Lambda \mathbf{x} = \mathbf{0}_\Lambda$

$\Rightarrow$ **Cosparsity**: $l := M - ||\mathbf{\Omega x}||_0$ [Nam et al, '13]
  - subspace: $W_\Lambda := N(\mathbf{\Psi}_\Lambda \mathbf{\Omega})$

- In the non-singular case, the two are equivalent: $\mathbf{\Omega}^{-1} = \mathbf{D}$
- **Prior Work**: [Elad et al, '07], [Nam et al, '13], for full-rank operators; in general the two models are not equivalent
- We consider square rank-deficient (difference) operators in the structured domain of graphs with $\mathbf{\Omega} = \mathbf{L}$ and $\mathbf{D} = \mathbf{L}^\dagger$ as the Moore-Penrose Pseudoinverse (MPP)

$\Rightarrow$ Characterize the underlying subspaces to understand how the models are fundamentally interrelated & uncover transitional properties

Matrix $\mathbf{\Psi}_\Lambda$ selects the rows of $\mathbf{\Omega}$ in set Λ

The Cosparse Analysis Model on Graphs

Prop. 1

The analysis subspace $W_\Lambda := N(\mathbf{\Psi}_\Lambda \mathbf{L})$ on a connected graph $G = (V, E)$ is given by

$$N(\mathbf{\Psi}_\Lambda \mathbf{L}) = z\mathbf{1}_N + \mathbf{L}^\dagger \mathbf{\Psi}_{\Lambda^c}^T \mathbf{W}\mathbf{c},$$

where $\mathbf{W} \in \mathbb{R}^{|\Lambda^c| \times |\Lambda^c|-1}$

- $$\mathbf{W} := \begin{pmatrix} |\Lambda^c|-1 & 0 & & \dots & 0 \\ -1 & |\Lambda^c|-2 & 0 & \dots & 0 \\ & -1 & |\Lambda^c|-3 & & \vdots \\ \vdots & & & & \\ & & & & 0 \\ & & & & 1 \\ -1 & -1 & \dots & & -1 \end{pmatrix}.$$

for $z \in \mathbb{R}$, $\mathbf{c} \in \mathbb{R}^{|\Lambda^c|-1}$.

- We require the constraint $\mathbf{W}$ s.t.$\mathbf{\Psi}_{\Lambda^c}^T \mathbf{W}\mathbf{c} \perp \mathbf{1}_N$ (**Fredholm Alternative**) on the solution subspaces
- $N(\mathbf{\Psi}_\Lambda \mathbf{L})$ has rank $N - |\Lambda| = |\Lambda^c|$ for $|\Lambda| < N$
- The subspace $\mathbf{L}^\dagger \mathbf{\Psi}_{\Lambda^c}^T \mathbf{W}$ is empty for $|\Lambda| \geq N - 1$.

Analysis Constraints

- The constraint $\mathbf{W}$ has a zero-sum column structure, facilitating

$$\mathbf{\Psi}_{\Lambda^c}^T \mathbf{W}\mathbf{c} = \mathbf{S}^T \mathbf{t}$$

  for suitable $\mathbf{c} \in \mathbb{R}^{|\Lambda^c|-1}$ and $\mathbf{t} \in \mathbb{R}^{|E|}$

- In general, any basis in $(\mathbf{e}_i - \mathbf{e}_j),\ i, j \in \Lambda^c \subset V$, is acceptable, where $e_i(i) = 1,\ e_i(j) = 0,\ j \neq i$

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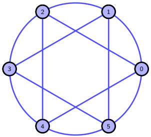
- Example

$$\mathbf{S}^T = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & -1 \end{pmatrix}$$



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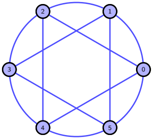
$$\mathbf{S}^T = \begin{pmatrix} \overset{E_0}{1} & 1 & 1 & 1 & \overset{E_4}{0} & 0 & 0 & \overset{E_7}{0} & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & -1 \end{pmatrix}$$



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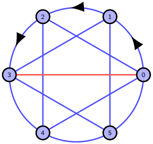

$$\mathbf{E}_0 + \mathbf{E}_4 + \mathbf{E}_7 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{pmatrix} \notin \mathbf{E}$$

The Sparse Synthesis Model on Graphs

- Consider $\mathbf{x} = \mathbf{L}^{\dagger}\mathbf{c}$ on connected G, with $\mathbf{D} = \mathbf{L}^{\dagger}$ and sparse $\mathbf{c} \in \mathbb{R}^N$ of support $\Lambda^c \subset V$
- The MPP $\mathbf{L}^{\dagger}$, with $\mathbf{L}\mathbf{L}^{\dagger} = \mathbf{I}_N - \frac{1}{N}\mathbf{J}_N$ and $\mathbf{L}^{\dagger}\mathbf{1}_N = \mathbf{0}_N$, is the **discrete Green's function** of $\mathbf{L}$

- We have $\mathbf{L}(\mathbf{L}^{\dagger}\mathbf{S}^T) = \mathbf{L}(\mathbf{S}^{\dagger}) = \mathbf{S}^T$

⇒ Any piecewise smooth signal on G is at least 2-sparse w.r.t $\mathbf{L}$, in the range of $\mathbf{S}^T$

- The analysis operation $\mathbf{L}\mathbf{x} = \mathbf{c}$ characterizes the constrained synthesis representation

$$\mathbf{x} = \mathbf{L}^{\dagger} \sum_{j \in E_S} \mathbf{S}_j^T = \sum_{j \in E_S} \mathbf{S}_j^{\dagger}, \text{ with } \mathbf{c} = \sum_{j \in E_S} \mathbf{S}_j^T$$

- The functions $\mathbf{L}^{\dagger}$ & $\mathbf{S}^{\dagger}$ encapsulate different orders of **smoothness** and **hop-localization** w.r.t. operators $\mathbf{L}^2$ & $\mathbf{L}$:

$$\mathbf{L}^2\mathbf{L}^{\dagger} = \mathbf{L} \text{ and } \mathbf{L}\mathbf{S}^{\dagger} = \mathbf{S}^T$$

  and the locations of non-zeros in the range of $\mathbf{L}$ and $\mathbf{S}^T$ can be interpreted as its 'knots'

⇒ The Gram structure of $\mathbf{L}$ with sparse $\mathbf{S}^T$ reveals an underlying **structured sparsity** on graphs

Union of Subspaces Model - Comparison

Thm. 1

- On a connected graph, the cosparse analysis model, $N(\mathbf{\Psi}_\Lambda \mathbf{L}) = span(\mathbf{1}_N; \mathbf{L}^\dagger(\mathbf{e}_i - \mathbf{e}_j), i, j \in \Lambda^c)$, is a **constrained** instance of the sparse synthesis model, $span(\mathbf{L}_j^\dagger, j \in \Lambda^c)$, up to a **translation** by $N(\mathbf{L}) = \mathbf{1}_N$.

- Signals $\mathbf{x}$ which satisfy $||\mathbf{L}\mathbf{x}||_0 = N - l$ (or $\mathbf{L}_\Lambda \mathbf{x} = \mathbf{0}_\Lambda$) are in the **analysis UoS** of cardinality $|\Lambda| = l$

$$\bigcup_{|\Lambda|=l} W_\Lambda, \text{ for } W_\Lambda := N(\mathbf{L}_\Lambda)$$

- Signals $\mathbf{x}$ which satisfy $\mathbf{x} = \mathbf{L}^\dagger \mathbf{c}$ with $||\mathbf{c}||_0 = k$ (or $\mathbf{x} = \mathbf{L}^\dagger_{\Lambda^c} \mathbf{c}_{\Lambda^c}$) are in the **synthesis UoS** of cardinality $|\Lambda^c| = k$

$$\bigcup_{|\Lambda^c|=k} V_{\Lambda^c}, \text{ for } V_{\Lambda^c} := span(\mathbf{L}_j^\dagger, j \in \Lambda^c)$$

	Synthesis			Analysis		
Sparsity	Dim.	Subsp.	No.	Dim.	Subsp.	No.
1	1	$\mathbf{L}_j^\dagger$	N	1	$\mathbf{1}_N$	1
2	2	$span(\mathbf{L}_j^\dagger, j \in \Lambda^c)$	$\binom{N}{2}$	2	$span(\mathbf{1}_N; \mathbf{L}^\dagger(\mathbf{e}_i - \mathbf{e}_j), i, j \in \Lambda^c)$	$\binom{N}{2}$
$k \ll N$	k	$span(\mathbf{L}_j^\dagger, j \in \Lambda^c)$	$\binom{N}{k}$	k	$span(\mathbf{1}_N; \mathbf{L}^\dagger(\mathbf{e}_i - \mathbf{e}_j), i, j \in \Lambda^c)$	$\binom{N}{k}$

Table 1: Subspace Characterization of **L** for a Connected Graph

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$$\bigcup_{|\Lambda|=l} W_\Lambda, \quad \text{for} \quad W_\Lambda := N(\mathbf{L}_\Lambda)$$

- Signals $\mathbf{x}$ which satisfy $\mathbf{x} = \mathbf{L}^\dagger \mathbf{c}$ with $||\mathbf{c}||_0 = k$ (or $\mathbf{x} = \mathbf{L}^\dagger_{\Lambda^c} \mathbf{c}_{\Lambda^c}$) are in the **synthesis UoS** of cardinality $|\Lambda^c| = k$

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$k \ll N$	k	$span(\mathbf{L}_j^\dagger, j \in \Lambda^c)$	$\binom{N}{k}$	k	$span(\mathbf{1}_N; \mathbf{L}^\dagger(\mathbf{e}_i - \mathbf{e}_j), i, j \in \Lambda^c)$	$\binom{N}{k}$

Table 1: Subspace Characterization of **L** for a Connected Graph

- If $N(\mathbf{L})$ is omitted, W_Λ has dimension $k-1$ and $\bigcup_{|\Lambda|=N-k} W_\Lambda \subseteq \bigcup_{|\Lambda^c|=k} V_{\Lambda^c}$

Union of Subspaces Model: Disconnected Graph

- $G = (V, E)$ has t connected components C_k s.t. $V = \bigcup_{k=1}^{t} C_k$, with $|C_k| = N_k$
- $N(\boldsymbol{\Psi}_\Lambda \mathbf{L})$ is given as the span of

$$\mathbf{L}^\dagger \boldsymbol{\Psi}_{\Lambda^c}^T \mathbf{W} = \begin{bmatrix} \mathbf{L}_1^\dagger \tilde{\boldsymbol{\Psi}}_{\Lambda_1^c}^T \mathbf{W}_1 & 0 & \dots & \\ 0 & \mathbf{L}_2^\dagger \tilde{\boldsymbol{\Psi}}_{\Lambda_2^c}^T \mathbf{W}_2 & 0 & \dots \\ \dots & & & \\ 0 & \dots & & \mathbf{L}_t^\dagger \tilde{\boldsymbol{\Psi}}_{\Lambda_t^c}^T \mathbf{W}_t \end{bmatrix}, \text{ with } \tilde{\boldsymbol{\Psi}}_{\Lambda_k} \in \mathbb{R}^{|\Lambda_k| \times N_k} \text{ and } C_k = \Lambda_k \cup \Lambda_k^c$$

of rank at least $|\Lambda^c| - t$, where $\mathbf{1}_{N_k}^T \tilde{\boldsymbol{\Psi}}_{\Lambda_k^c}^T \mathbf{W}_k = 0$, and $N(\mathbf{L}) = \{\mathbf{1}_{C_1}, \dots, \mathbf{1}_{C_t}\}$ of rank t

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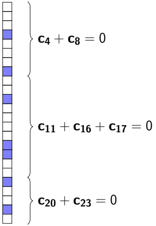
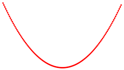
- The constraints form a **Structured Sparsity Model** with blocks (components) C_k, whose coefficients respectively sum to 0

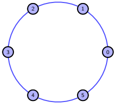


- For $k = |\Lambda^c|$ with $k < N_i$, the **synthesis UoS** has $\binom{N}{k}$ subspaces V_{Λ^c} of dimension k, while the **analysis UoS** has $L < \binom{N}{k}$ subspaces W_Λ of dimensions ranging from k to $k + t - 1$

$\Rightarrow$ The dimension & number of analysis subspaces become non-uniform

The Simple Cycle





- On the simple cycle G_C, the rows (columns) of $\mathbf{L}_C$ have 2 vanishing moments [MSK, '17] & $\mathbf{L}_C^\dagger$ has entries $L_C^\dagger(i,j) = \frac{(N-1)(N+1)}{12N} - \frac{1}{2}|j-i| + \frac{(j-i)^2}{2N}$, for $0 \leq i,j \leq N-1$ [Ellis, '03]

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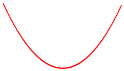

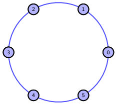

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- Differences $\mathbf{L}_C^\dagger(\mathbf{e}_i - \mathbf{e}_j)$, $i,j \in V$, are piecewise linear

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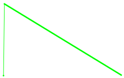

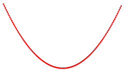

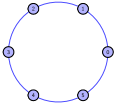

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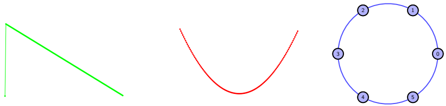
- Differences $\mathbf{L}_C^\dagger(\mathbf{e}_i - \mathbf{e}_j),\ i,j \in V$, are piecewise linear
- $\mathbf{S}_C^\dagger$ of circulant incidence matrix $\mathbf{S}_C$ with first row $[1\ \ -1\ 0\ \dots\ 0]$ has entries

$$S_C^\dagger(i,j) = \frac{N-1}{2N} - \frac{j-i}{N},\ \text{for}\ i \leq j, \quad 0 \leq i,j \leq N-1$$

The Simple Cycle



- On the simple cycle G_C, the rows (columns) of $\mathbf{L}_C$ have 2 vanishing moments [MSK, '17] & $\mathbf{L}_C^{\dagger}$ has entries $L_C^{\dagger}(i,j) = \frac{(N-1)(N+1)}{12N} - \frac{1}{2}|j-i| + \frac{(j-i)^2}{2N}$, for $0 \leq i,j \leq N-1$ [Ellis, '03]
- Differences $\mathbf{L}_C^{\dagger}(\mathbf{e}_i - \mathbf{e}_j),\ i,j \in V$, are piecewise linear
- $\mathbf{S}_C^{\dagger}$ of circulant incidence matrix $\mathbf{S}_C$ with first row $[1\ \ -1\ 0\ \dots\ 0]$ has entries

$$S_C^{\dagger}(i,j) = \frac{N-1}{2N} - \frac{j-i}{N}, \text{ for } i \leq j, \quad 0 \leq i,j \leq N-1$$

- The sparse synthesis model on G_C, $span((\mathbf{L}_C^{\dagger})_j,\ j \in \Lambda^c)$, generates up to **piecewise quadratic polynomials**, orthogonal to $\mathbf{1}_N$
- The cosparse analysis model on G_C, defined by

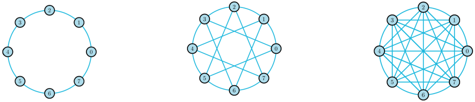
$$N(\mathbf{\Psi}_{\Lambda}\mathbf{L}_C) = span(\mathbf{1}_N; \mathbf{L}_C^{\dagger}(\mathbf{e}_i - \mathbf{e}_j),\ i,j \in \Lambda^c), \text{ with } \mathbf{L}_C^{\dagger}(\mathbf{e}_i - \mathbf{e}_j) = \sum_{k \in E_S} t_k (\mathbf{S}_C^{\dagger})_k$$

  generates up to **piecewise linear polynomials**, for suitable edge sequence $E_S \subset E$, and $t_k \in \mathbb{R}$.

$\Rightarrow$ broad representation range w.r.t. both $\mathbf{L}_C^{\dagger}$ and $\mathbf{S}_C^{\dagger}$

$\Rightarrow$ **synthesis interpretation** of (classical) vanishing moment constraints

General Circulant Graphs

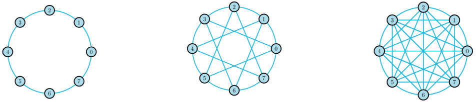


(a) $S = \{1\}$ (b) $S = \{1, 3\}$ (c) $S = \{1, 2, 3, 4\}$

$\Rightarrow$ Models for general circulant graphs can be developed on the basis of the simple cycle:

- A graph G_S is circulant w.r.t. generating set $S = \{s_i\}_{i=1}^{M}$, $0 < s_k \leq N/2$, if nodes $(i, (i \pm s_k)_N)$ are adjacent, $\forall s_k \in S \Rightarrow G_S$ is circulant if $\mathbf{L}$ is circulant

General Circulant Graphs



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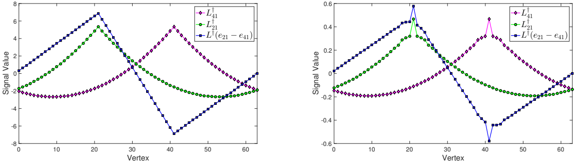
- **Lemma:** On connected G_S, with $s = 1 \in S$ & bandwidth $M < N/2$, we can decompose $\mathbf{L}$ as $\boxed{\mathbf{L} = \mathbf{P}_{G_S}\mathbf{L}_C}$, where $\mathbf{P}_{G_S}$ is circulant positive definite of bandwidth $M - 1$.
- $\mathbf{P}_{G_S}$ encapsulates the connectivity information of G_S
- We have $\mathbf{L}^\dagger = \mathbf{P}_{G_S}^{-1}\mathbf{L}_C^\dagger$, where the entries of $\mathbf{P}_{G_S}^{-1}$ exhibit exponential decay (in absolute value), 'perturbing' $\mathbf{L}_C^\dagger$
- The columns of $\mathbf{S}^\dagger$ are 'perturbed' piecewise linear polynomials

Thm. 2

- The cosparse analysis model on circulant graphs generates perturbed piecewise linear polynomials $N(\mathbf{\Psi}_\Lambda \mathbf{L}) = z\mathbf{1}_N + \mathbf{P}_{G_S}^{-1}\mathbf{L}_C^{\dagger}\mathbf{\Psi}_{\Lambda^c}^T\mathbf{W}\mathbf{c}_1$, $\mathbf{c}_1 \in \mathbb{R}^{|\Lambda^c|-1}$, which are translated by $\mathbf{1}_N$, while the sparse synthesis model generates perturbed piecewise quadratic polynomials, $\mathbf{P}_{G_S}^{-1}\mathbf{L}_C^{\dagger}\mathbf{\Psi}_{\Lambda^c}^T\mathbf{c}_2$, $\mathbf{c}_2 \in \mathbb{R}^{|\Lambda^c|}$.

$\Rightarrow$ The analysis constraint reduces the order of the functions which define its subspaces



(a) Functions on G_S with $S = \{1\}$ (b) Functions on G_S with $S = \{1, 2, 3\}$

Figure 1: Comparison of signal models on circulant graphs

The Generalized Graph Laplacian

- Parametric $\mathbf{L}_\alpha = d_\alpha \mathbf{I}_N - \mathbf{A}$, with $d_\alpha = \sum_{j=1}^{M} 2d_j \cos(\alpha j)$, $\alpha \in \mathbb{C}$, and weights $d_j = A_{i,(j+i)_N}$, annihilates $\mathbf{x} = e^{\pm i\alpha \mathbf{t}}$, $\mathbf{t} = [0 \ldots N-1]$ on circulant graphs of bandwidth M [MSK, '17]
- On the simple cycle, the rows (columns) of $\mathbf{L}_{C,\alpha}$ have 2 exponential vanishing moments
- $\mathbf{L}_{C,\alpha}$ is singular for $\alpha = 2\pi k/N, \; k \in \mathbb{N}$, with $N(\mathbf{L}_{C,\alpha}) = span(e^{i\alpha\mathbf{t}}, e^{-i\alpha\mathbf{t}})$, & non-singular o/w

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- **Lemma 1** For $\alpha \neq 2\pi k/N,\ \ k \in \mathbb{N}$, $\mathbf{L}_{C,\alpha}^{-1}$ has entries

$$L_{C,\alpha}^{-1}(m,n) = \frac{1}{(-e^{-i\alpha} + e^{i\alpha})(-1 + e^{i\alpha N})} \boxed{e^{i\alpha|n-m|}} + \frac{1}{(e^{-i\alpha} - e^{i\alpha})(-1 + e^{-i\alpha N})} \boxed{e^{-i\alpha|n-m|}},$$
$$0 \leq m,n \leq N-1.$$

$\Rightarrow$ The rows (columns) of $\mathbf{L}_{C,\alpha}^{-1}$ are **complex exponentials**

- **Lemma 2** For $\alpha = 2\pi k/N,\ \ k \in \mathbb{N}$ and $\alpha \neq 0, k\pi$, $\mathbf{L}_{C,\alpha}^{\dagger}$ has entries

$$L_{C,\alpha}^{\dagger}(m,n) = \frac{e^{i\alpha}}{2N} \left(\frac{\boxed{2|n-m|}(-1 + e^{2i\alpha}) + (N-1) - e^{2i\alpha}(N+1)}{(-1 + e^{2i\alpha})^2} \right) \boxed{e^{i\alpha|n-m|}}$$
$$+ \frac{e^{-i\alpha}}{2N} \left(\frac{\boxed{2|n-m|}(-1 + e^{-2i\alpha}) + (N-1) - e^{-2i\alpha}(N+1)}{(-1 + e^{-2i\alpha})^2} \right) \boxed{e^{-i\alpha|n-m|}},\ \ 0 \leq m,n \leq N-1.$$

$\Rightarrow$ The rows (columns) of $\mathbf{L}_{C,\alpha}^{\dagger}$ are **linear complex exponential polynomials**

The Generalized Graph Laplacian

- On general circulant graphs G_S, we have $\mathbf{L}_\alpha = \mathbf{L}_{C,\alpha}\mathbf{P}_\alpha$, where $\mathbf{P}_\alpha$ is circulant of bandwidth $M-1$ and depends on the graph connectivity
- $\mathbf{P}_\alpha$ is positive definite up to certain $\alpha \in \mathbb{C}$ and G_S, then $\mathbf{P}_\alpha^{-1}$ invokes a localized perturbation
- **Thm. 3**

  For $\alpha \neq 2\pi k/N,\ \ k \in \mathbb{N}$, the cosparse analysis and sparse synthesis models of $\mathbf{L}_\alpha$ are **equivalent**, generating **perturbed complex exponentials**

  $$\mathbf{P}_\alpha^{-1}\mathbf{L}_{C,\alpha}^{-1}\mathbf{\Psi}_{\Lambda^c}^T,\ \Lambda^c \subset V.$$

  For $\alpha = 2\pi k/N,\ \alpha \neq 0, k\pi,\ \ k \in \mathbb{N}$, the sparse synthesis model generates **perturbed linear complex exponential polynomials**

  $$\mathbf{P}_\alpha^{-1}\mathbf{L}_{C,\alpha}^{\dagger}\mathbf{\Psi}_{\Lambda^c}^T,\ \Lambda^c \subset V.$$

  The cosparse analysis model generates the **constrained, translated** subspaces

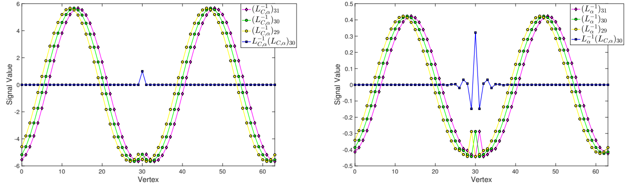
  $$N(\mathbf{L}_\alpha) + \mathbf{P}_\alpha^{-1}\mathbf{L}_{C,\alpha}^{\dagger}\mathbf{\Psi}_{\Lambda^c}^T\mathbf{W}_\alpha\mathbf{c},\ \Lambda^c \subset V,$$

  for constraint $\mathbf{W}_\alpha \in \mathbb{C}^{|\Lambda^c|\times|\Lambda^c|-2}$ such that $\mathbf{\Psi}_{\Lambda^c}^T(\mathbf{W}_\alpha)_j \perp e^{\pm i\alpha\mathbf{t}}$.

  If $\mathbf{\Psi}_{\Lambda^c}^T\mathbf{W}_\alpha\mathbf{c} = (\mathbf{L}_{C,\alpha})_j$ for some $j \in V$, this reduces to **perturbed complex exponentials**

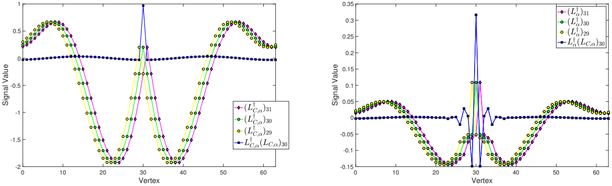
  $$N(\mathbf{L}_\alpha) + \mathbf{P}_\alpha^{-1}\left(\mathbf{I}_N - \frac{1}{N}\mathbf{E}_\alpha\right)\tilde{\mathbf{c}},\ \text{for suitable } \tilde{\mathbf{c}} \in \mathbb{C}^N$$

  where $\mathbf{E}_\alpha$ is the projection onto $N(\mathbf{L}_\alpha)$, and is **comparable in order** to the case $\alpha \neq 2\pi k/N,\ k \in \mathbb{N}$.
- For $\alpha = 0$, this reduces to the graph Laplacian $\mathbf{L}$



(a) G_S with $S = \{1\}$ (b) G_S with $S = \{1, 2, 3\}$

Figure 2: Comparison of signal models on circulant graphs for $\mathbf{L}_{\alpha}^{-1}$, $\alpha = 0.21$.



(a) G_S with $S = \{1\}$ (b) G_S with $S = \{1, 2, 3\}$

Figure 3: Comparison of signal models on circulant graphs for $\mathbf{L}_{\alpha}^{\dagger}$, $\alpha = 4\pi / N$.

Uniqueness Guarantees

- Suppose $\mathbf{x}$ belongs to a graph Laplacian based UoS model on an undirected graph:

$\Rightarrow$ Identify the unique (co)sparse solution of $\mathbf{y} = \mathbf{M}\mathbf{x}$, for suitable $\mathbf{M} \in \mathbb{R}^{m \times N}$, $m < N$, with linearly independent rows

- Given mutually independent $\mathbf{M}$ and $\boldsymbol{\Omega}$, we require $m \geq \tilde{\kappa}_{\boldsymbol{\Omega}}(l)$, with

$$\tilde{\kappa}_{\boldsymbol{\Omega}}(l) := \max\{dim(W_{\Lambda_1} + W_{\Lambda_2}) : \ |\Lambda_i| \geq l, i = 1, 2\}$$

to uniquely identify $\mathbf{x}$ with $\boldsymbol{\Omega}_{\Lambda}\mathbf{x} = \mathbf{0}_{\Lambda}$, $l = N - ||\boldsymbol{\Omega}\mathbf{x}||_0$ [Lu et al, '07]

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to uniquely identify $\mathbf{x}$ with $\mathbf{\Omega}_\Lambda \mathbf{x} = \mathbf{0}_\Lambda$, $l = N - ||\mathbf{\Omega}\mathbf{x}||_0$ [Lu et al, '07]

- **Corollary:** For mutually independent $\mathbf{M} \in \mathbb{R}^{m \times N}$ and $\mathbf{\Omega} = \mathbf{L}$ on $G = (V, E)$, the problem

$$\mathbf{M}\mathbf{x} = \mathbf{y} \text{ with } ||\mathbf{L}\mathbf{x}||_0 \leq N - l = k$$

has at most one solution, provided $k > 1$, if

(i) $\boxed{m \geq 2k - 1}$, when the graph is connected,

$(ii1)$ $\boxed{m \geq 2k - 2 + c}$, when the graph is disconnected with c components.

$(ii2)$ If $\mathbf{x} \in \bigcup_{|\Lambda|=l} W_\Lambda$, for $W_\Lambda := N(\mathbf{L}_\Lambda)$, subject to $|\Lambda_i| < N_i - 1$, $(ii1)$ becomes $\boxed{m \geq 2k - c}$.

- For a stable sampling scheme, m necessarily depends on $ln(L)$ and K, for L total subspaces with maximum dimension K in a UoS [Blumensath et al, '09]

$\Rightarrow$ Model-based Compressed Sensing (on Graphs)

Conclusion and Future Work

- We have substantiated the discrepancy between the cosparse analysis and sparse synthesis models for the graph Laplacian through subspace analysis
- We have characterized the functions defining the respective model subspaces on circulant graphs
- For the parametric graph Laplacian on circulant graphs, we have shown transitional properties between model equivalence and non-equivalence

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- We have characterized the functions defining the respective model subspaces on circulant graphs
- For the parametric graph Laplacian on circulant graphs, we have shown transitional properties between model equivalence and non-equivalence

⇒ Develop refined UoS signal models on graphs with enhanced sampling schemes & recovery guarantees

For a comprehensive discussion, refer to arXiv

Analysis vs Synthesis with Structure - An Investigation of Union of Subspace Models on Graphs

https://arxiv.org/abs/1811.04493

Thank you.

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