Analysis vs Synthesis -
An Investigation of (Co)sparse Signal Models on Graphs

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GlobalSIP 2018

Anaheim, Nov 2018
Motivation and Objectives

- Characterize Sparsity on Graphs - w.r.t. the graph connectivity & defining subspaces
- Signal Models: Tackle Analysis vs Synthesis Problem in the structured setting of graphs
- Establish discrepancy between Analysis & Synthesis view of graph Laplacian through subspace analysis
- For circulant graphs
- Develop closed-form expressions of functions defining the subspaces & concretize discrepancy
- Transition between model equivalence and non-equivalence for the parametric graph Laplacian
- Unify results to quantify uniqueness guarantees for signals in UoS models on graphs

⇒ Links between Graph Theory, PDEs & Linear Algebra render problem investigation feasible
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A graph $G = (V, E)$ is defined by a vertex set $V = \{0, \ldots, N-1\}$, with $|V| = N$, and edge set $E = \{E_0, \ldots, E_{M-1}\}$.

The adjacency matrix $A$ captures the connectivity of $G$, with

$$A_{i,j} > 0, \text{ if } i \text{ and } j \text{ are adjacent } (i \neq j), \quad A_{i,j} = 0, \text{ otherwise}$$

and $D$ is the diagonal degree matrix with $D_{i,i} = \sum_j A_{i,j}$.

The non-normalized graph Laplacian is given by $L = D - A$.

The oriented incidence matrix $S \in \mathbb{R}^{|E| \times |V|}$ has entries

$$S_{k,i} = \sqrt{A_{i,j}}, \quad S_{k,j} = -\sqrt{A_{i,j}}, \text{ if edge } E_k = \{i, j\} \text{ is directed as } i \to j$$

and we have $L = S^T S$.

We consider undirected, and (un-)weighted graphs without self-loops.

The graph signal $x$ on $G$, with $x : V \to \mathbb{C}$ s.t. $x(i)$ is the sample value of $x \in \mathbb{C}^N$ at vertex $i$, is piecewise smooth w.r.t. $L$ if $Lx$ is sparse, i.e. $\|Lx\|_0 \ll N$.
The Analysis vs Synthesis Problem

**Synthesis**

- generate signal $\mathbf{x} = \mathbf{Dc}$, given dictionary $\mathbf{D} \in \mathbb{R}^{N \times M}$, $N \leq M$, and $\mathbf{c} \in \mathbb{R}^M$ with $||\mathbf{c}||_0 = k \ll M$ of sparse support $\Lambda^c$
- subspace: $V_{\Lambda^c} := \text{span}(\mathbf{D}_j, j \in \Lambda^c)$

**Analysis**

- given analysis operator $\Omega \in \mathbb{R}^{M \times N}$, apply constraint $||\Omega \mathbf{x}||_0 = k \ll M$ with $\Omega_{\Lambda^c} \mathbf{x} = \mathbf{0}_{\Lambda^c}$
- **Cosparsity**: $l := M - ||\Omega \mathbf{x}||_0$ [Nam et al, '13]
- subspace: $W_{\Lambda} := \mathcal{N}(\Psi_{\Lambda} \Omega)$

- In the non-singular case, the two are equivalent: $\Omega^{-1} = \mathbf{D}$

- **Prior Work**: [Elad et al, '07], [Nam et al, '13], for full-rank operators; in general the two models are not equivalent

- We consider square rank-deficient (difference) operators in the structured domain of graphs with $\Omega = \mathbf{L}$ and $\mathbf{D} = \mathbf{L}^\dagger$ as the Moore-Penrose Pseudoinverse (MPP)

- Characterize the underlying subspaces to understand how the models are fundamentally interrelated & uncover transitional properties

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Matrix $\Psi_{\Lambda}$ selects the rows of $\Omega$ in set $\Lambda$
Prop. 1

The analysis subspace $W_{\Lambda} := N(\Psi_{\Lambda}L)$ on a connected graph $G = (V, E)$ is given by

$$N(\Psi_{\Lambda}L) = z1_N + L^\dagger \Psi_{\Lambda_c}^T W_c,$$

where $W \in \mathbb{R}^{|\Lambda_c| \times |\Lambda_c| - 1}$

$$W := \begin{pmatrix}
|\Lambda_c| - 1 & 0 & 0 & \cdots & 0 \\
-1 & |\Lambda_c| - 2 & 0 & \cdots & 0 \\
& -1 & |\Lambda_c| - 3 & \cdots & \\
& & & \ddots & \\
& & & & -1
\end{pmatrix},$$

for $z \in \mathbb{R}$, $c \in \mathbb{R}^{|\Lambda_c| - 1}$.

- We require the constraint $W$ s.t. $\Psi_{\Lambda_c}^T W c \perp 1_N$ (Fredholm Alternative) on the solution subspaces.
- $N(\Psi_{\Lambda}L)$ has rank $N - |\Lambda| = |\Lambda_c|$ for $|\Lambda| < N$.
- The subspace $L^\dagger \Psi_{\Lambda_c}^T W$ is empty for $|\Lambda| \geq N - 1$. 
Analysis Constraints

- The constraint $W$ has a zero-sum column structure, facilitating

$$\Psi_{\Lambda^c}^T Wc = S^T t$$

for suitable $c \in \mathbb{R}^{|\Lambda^c|}^{-1}$ and $t \in \mathbb{R}^{|E|}$

- In general, any basis in $(e_i - e_j), \ i, j \in \Lambda^c \subset V$, is acceptable, where $e_i(i) = 1, \ e_i(j) = 0, \ j \neq i$
Analysis Constraints

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  \[ \Psi_{\Lambda^c}^T Wc = S^T t \]
  for suitable $c \in \mathbb{R}^{|\Lambda^c| - 1}$ and $t \in \mathbb{R}^{|E|}$.

- In general, any basis in $(e_i - e_j)$, $i, j \in \Lambda^c \subset V$, is acceptable, where $e_i(i) = 1$, $e_i(j) = 0$, $j \neq i$.

- Example

\[
S^T = \begin{pmatrix}
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 1 & 1 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 1 \\
0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & -1
\end{pmatrix}
\]
The constraint $W$ has a zero-sum column structure, facilitating

$$\Psi_{\Lambda^c}^T Wc = S^T t$$

for suitable $c \in \mathbb{R}^{|\Lambda^c|-1}$ and $t \in \mathbb{R}^{|E|}$

In general: basis in $(e_i - e_j), \ i, j \in \Lambda^c \subset V$, is acceptable, where $e_i(i) = 1, \ e_i(j) = 0, \ j \neq i$

Example

$$S^T = \begin{pmatrix}
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0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 1 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 \\
\end{pmatrix}$$
Analysis Constraints

▶ The constraint $W$ has a zero-sum column structure, facilitating

$$\Psi^T_{\Lambda^c} W c = S^T t$$

for suitable $c \in \mathbb{R}^{|\Lambda^c|^{-1}}$ and $t \in \mathbb{R}^{|E|}$

▶ In general: basis in $(e_i - e_j)$, $i, j \in \Lambda^c \subset V$, is acceptable, where $e_i(i) = 1$, $e_i(j) = 0$, $j \neq i$

▶ Example

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S^T = \begin{pmatrix}
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-1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 1 & 1 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & -1
\end{pmatrix}
\]

\[
E_0 + E_4 + E_7 = \begin{pmatrix}
1 \\
0 \\
0 \\
-1 \\
0 \\
0
\end{pmatrix} \notin E
\]
The Sparse Synthesis Model on Graphs

Consider \( x = L^\dagger c \) on connected \( G \), with \( D = L^\dagger \) and sparse \( c \in \mathbb{R}^N \) of support \( \Lambda^c \subset V \)

The MPP \( L^\dagger \), with \( LL^\dagger = I_N - \frac{1}{N} J_N \) and \( L^\dagger 1_N = 0_N \), is the discrete Green's function of \( L \)

We have \( L(L^\dagger S^T) = L(S^\dagger) = S^T \)

Any piecewise smooth signal on \( G \) is at least 2-sparse w.r.t \( L \), in the range of \( S^T \)

The analysis operation \( Lx = c \) characterizes the constrained synthesis representation

\[
x = L^\dagger \sum_{j \in E_S} S_j^T = \sum_{j \in E_S} S_j^\dagger, \quad \text{with} \quad c = \sum_{j \in E_S} S_j^T
\]

The functions \( L^\dagger \) & \( S^\dagger \) encapsulate different orders of smoothness and hop-localization w.r.t. operators \( L^2 \) & \( L \):

\[
L^2 L^\dagger = L \quad \text{and} \quad LS^\dagger = S^T
\]

and the locations of non-zeros in the range of \( L \) and \( S^T \) can be interpreted as its ‘knots’

The Gram structure of \( L \) with sparse \( S^T \) reveals an underlying structured sparsity on graphs
Thm. 1

On a connected graph, the cosparse analysis model, \( N(\Psi_{\Lambda}L) = \text{span}(1_N; L^\dagger(e_i - e_j), i, j \in \Lambda^c) \), is a constrained instance of the sparse synthesis model, \( \text{span}(L_j^\dagger, j \in \Lambda^c) \), up to a translation by \( N(L) = 1_N \).

Signals \( x \) which satisfy \( ||Lx||_0 = N - l \) (or \( L_{\Lambda}x = 0_{\Lambda} \)) are in the analysis UoS of cardinality \( |\Lambda| = l \)
\[
\bigcup_{|\Lambda| = l} W_\Lambda, \text{ for } W_\Lambda := N(L_{\Lambda})
\]

Signals \( x \) which satisfy \( x = L^\dagger c \) with \( ||c||_0 = k \) (or \( x = L^\dagger_{\Lambda^c} c_{\Lambda^c} \)) are in the synthesis UoS of cardinality \( |\Lambda^c| = k \)
\[
\bigcup_{|\Lambda^c| = k} V_{\Lambda^c}, \text{ for } V_{\Lambda^c} := \text{span}(L_j^\dagger, j \in \Lambda^c)
\]

<table>
<thead>
<tr>
<th>Sparsity</th>
<th>Synthesis</th>
<th>Analysis</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>( L_j^\dagger )</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>( \text{span}(L_j^\dagger, j \in \Lambda^c) )</td>
</tr>
<tr>
<td>( k \ll N )</td>
<td>( k )</td>
<td>( \text{span}(L_j^\dagger, j \in \Lambda^c) )</td>
</tr>
</tbody>
</table>

Table 1: Subspace Characterization of \( L \) for a Connected Graph
Thm. 1

On a connected graph, the cosparse analysis model, \( N(\Psi_\Lambda L) = \text{span}(1_N; L^\dagger (e_i - e_j), i, j \in \Lambda^c) \), is a **constrained** instance of the sparse synthesis model, \( \text{span}(L^\dagger_j, j \in \Lambda^c) \), up to a translation by \( N(L) = 1_N \).

Signals \( x \) which satisfy \( ||Lx||_0 = N - l \) (or \( L_\Lambda x = 0_\Lambda \)) are in the **analysis UoS** of cardinality \( |\Lambda| = l \)
\[
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Signals \( x \) which satisfy \( x = L^\dagger c \) with \( ||c||_0 = k \) (or \( x = L^\dagger_\Lambda c_{\Lambda^c} \)) are in the **synthesis UoS** of cardinality \( |\Lambda^c| = k \)
\[
\bigcup_{|\Lambda^c| = k} V_{\Lambda^c}, \text{ for } V_{\Lambda^c} := \text{span}(L^\dagger_j, j \in \Lambda^c)
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| 1        | 1          | N        | 1 \( \Lambda \)
| 2        | \( \text{span}(L^\dagger_j, j \in \Lambda^c) \) | \( \binom{N}{2} \) | \( \text{span}(1_N; L^\dagger (e_i - e_j), i, j \in \Lambda^c) \) | \( \binom{N}{k} \) |
| \( k \ll N \) | \( \text{span}(L^\dagger_j, j \in \Lambda^c) \) | \( \binom{N}{k} \) | \( \text{span}(1_N; L^\dagger (e_i - e_j), i, j \in \Lambda^c) \) | \( \binom{N}{k} \) |

*Table 1: Subspace Characterization of \( L \) for a Connected Graph*

If \( N(L) \) is omitted, \( W_\Lambda \) has dimension \( k - 1 \) and \( \bigcup_{|\Lambda| = N - k} W_\Lambda \subseteq \bigcup_{|\Lambda^c| = k} V_{\Lambda^c} \)
Union of Subspaces Model: Disconnected Graph

- $G = (V, E)$ has $t$ connected components $C_k$ s.t. $V = \bigcup_{k=1}^{t} C_k$, with $|C_k| = N_k$
- $N(\Psi_{\Lambda} L)$ is given as the span of

$$L^\dag \Psi_{\Lambda^c}^T W = \begin{bmatrix} L_1^\dag \tilde{\Psi}_{\Lambda_1^c}^T W_1 & 0 & \ldots \\ 0 & L_2^\dag \tilde{\Psi}_{\Lambda_2^c}^T W_2 & 0 & \ldots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & L_t^\dag \tilde{\Psi}_{\Lambda_t^c}^T W_t \end{bmatrix}, \text{ with } \tilde{\Psi}_{\Lambda_k} \in \mathbb{R}^{|\Lambda_k| \times N_k} \text{ and } C_k = \Lambda_k \cup \Lambda_k^c$$

of rank at least $|\Lambda^c| - t$, where $1_{N_k}^T \tilde{\Psi}_{\Lambda_k^c}^T W_k = 0$, and $N(L) = \{1_{C_1}, \ldots, 1_{C_t}\}$ of rank $t$
G = (V, E) has t connected components C_k s.t. V = \bigcup_{k=1}^{t} C_k, with |C_k| = N_k

N(\Psi_\Lambda L) is given as the span of

\[ L^\dagger \tilde{\Psi}_{\Lambda^c}^T W = \begin{bmatrix} L_1^\dagger \tilde{\Psi}_{\Lambda_1^c}^T W_1 & 0 & \ldots \\ 0 & L_2^\dagger \tilde{\Psi}_{\Lambda_2^c}^T W_2 & 0 & \ldots \\ \vdots \\ 0 & \ldots & L_t^\dagger \tilde{\Psi}_{\Lambda_t^c}^T W_t \end{bmatrix}, \text{ with } \tilde{\Psi}_{\Lambda_k} \in \mathbb{R}^{\Lambda_k \times N_k} \text{ and } C_k = \Lambda_k \cup \Lambda_k^c

\]
of rank at least |\Lambda^c| - t, where \( 1_{N_k}^T \tilde{\Psi}_{\Lambda_k^c}^T W_k = 0 \), and \( N(L) = \{1_{C_1}, \ldots, 1_{C_t}\} \) of rank t

The constraints form a **Structured Sparsity Model** with blocks (components) \( C_k \), whose coefficients respectively sum to 0

\[
\begin{align*}
&c_4 + c_8 = 0 \\
&c_{11} + c_{16} + c_{17} = 0 \\
&c_{20} + c_{23} = 0
\end{align*}
\]

For \( k = |\Lambda^c| \) with \( k < N_i \), the **synthesis UoS** has \( \binom{N}{k} \) subspaces \( V_{\Lambda^c} \) of dimension \( k \), while the **analysis UoS** has \( L < \binom{N}{k} \) subspaces \( W_\Lambda \) of dimensions ranging from \( k \) to \( k + t - 1 \)

⇒ The dimension & number of analysis subspaces become non-uniform
On the simple cycle $G_C$, the rows (columns) of $L_C$ have 2 vanishing moments [MSK, ’17] & $L_C^\dagger$ has entries $L_C^\dagger(i,j) = \frac{(N-1)(N+1)}{12N} - \frac{1}{2}|j-i| + \frac{(j-i)^2}{2N}$, for $0 \leq i,j \leq N-1$ [Ellis, '03].
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Differences $L_C^\dagger(e_i - e_j)$, $i, j \in V$, are piecewise linear
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$S_C^\dagger$ of circulant incidence matrix $S_C$ with first row $[1 \ 1 \ 0 \ldots \ 0]$ has entries

$$S_C^\dagger(i, j) = \frac{N-1}{2N} - \frac{i-j}{N}, \text{ for } i \leq j, \ 0 \leq i, j \leq N - 1$$
On the simple cycle $G_C$, the rows (columns) of $L_C$ have 2 vanishing moments [MSK, '17] & $L^+_C$ has entries $L^+_C(i, j) = \frac{(N-1)(N+1)}{12N} - \frac{1}{2}|j - i| + \frac{(j-i)^2}{2N}$, for $0 \leq i, j \leq N - 1$ [Ellis, '03]

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$S^+_C$ of circulant incidence matrix $S_C$ with first row $[1 - 1 0 \ldots 0]$ has entries $S^+_C(i, j) = \frac{N-1}{2N} - \frac{i-j}{N}$, for $i \leq j$, $0 \leq i, j \leq N - 1$

The sparse synthesis model on $G_C$, span($((L^+_C)_j, j \in \Lambda^c)$), generates up to piecewise quadratic polynomials, orthogonal to $1_N$

The cosparse analysis model on $G_C$, defined by $N(\Psi \Lambda L_C) = \text{span}(1_N; L^+_C(e_i - e_j), i, j \in \Lambda^c)$, with $L^+_C(e_i - e_j) = \sum_{k \in E_S} t_k (S^+_C)_k$

generates up to piecewise linear polynomials, for suitable edge sequence $E_S \subset E$, and $t_k \in \mathbb{R}$.

⇒ broad representation range w.r.t. both $L^+_C$ and $S^+_C$

⇒ synthesis interpretation of (classical) vanishing moment constraints
Models for general circulant graphs can be developed on the basis of the simple cycle:

- A graph $G_S$ is circulant w.r.t. generating set $S = \{s_i\}_{i=1}^{M}$, $0 < s_k \leq N/2$, if nodes $(i, (i \pm s_k) \mod N)$ are adjacent, $\forall s_k \in S \Rightarrow G_S$ is circulant if $L$ is circulant.
General Circulant Graphs

(a) $S = \{1\}$
(b) $S = \{1, 3\}$
(c) $S = \{1, 2, 3, 4\}$

⇒ Models for general circulant graphs can be developed on the basis of the simple cycle:

- A graph $G_S$ is circulant w.r.t. generating set $S = \{s_i\}_{i=1}^M$, $0 < s_k \leq N/2$, if nodes $(i, (i \pm s_k)_N)$ are adjacent, $\forall s_k \in S \Rightarrow G_S$ is circulant if $L$ is circulant

- **Lemma**: On connected $G_S$, with $s = 1 \in S$ & bandwidth $M < N/2$, we can decompose $L$ as
  \[ L = P_{G_S} L_C \]
  where $P_{G_S}$ is circulant positive definite of bandwidth $M - 1$.

- $P_{G_S}$ encapsulates the connectivity information of $G_S$

- We have $L^\dagger = P_{G_S}^{-1} L_C^\dagger$, where the entries of $P_{G_S}^{-1}$ exhibit exponential decay (in absolute value), ‘perturbing’ $L_C^\dagger$

- The columns of $S^\dagger$ are ‘perturbed’ piecewise linear polynomials
Thm. 2

The cosparse analysis model on circulant graphs generates perturbed piecewise linear polynomials
\[ N(\Psi \Lambda L) = z_1 N + P_{GS}^{-1} L_C^\dagger \Psi^T \Lambda_c W_c, \ c_1 \in \mathbb{R}^{|\Lambda_c| - 1}, \]
which are translated by \( 1_N \), while the sparse synthesis model generates perturbed piecewise quadratic polynomials,
\[ P_{GS}^{-1} L_C^\dagger \Psi^T \Lambda_c c_2, \ c_2 \in \mathbb{R}^{|\Lambda_c|}. \]

⇒ The analysis constraint reduces the order of the functions which define its subspaces.

(a) Functions on \( G_S \) with \( S = \{1\} \)
(b) Functions on \( G_S \) with \( S = \{1, 2, 3\} \)

Figure 1: Comparison of signal models on circulant graphs
The Generalized Graph Laplacian

- Parametric $\mathbf{L}_\alpha = d_\alpha \mathbf{I}_N - \mathbf{A}$, with $d_\alpha = \sum_{j=1}^{M} 2d_j \cos(\alpha j)$, $\alpha \in \mathbb{C}$, and weights $d_j = A_i, (j+i)N$, annihilates $x = e^{\pm i\alpha t}$, $t = [0 \ldots N - 1]$ on circulant graphs of bandwidth $M$ [MSK, '17]

- On the simple cycle, the rows (columns) of $\mathbf{L}_{C, \alpha}$ have 2 exponential vanishing moments

- $\mathbf{L}_{C, \alpha}$ is singular for $\alpha = 2\pi k / N$, $k \in \mathbb{N}$, with $N(\mathbf{L}_{C, \alpha}) = \text{span}(e^{i\alpha t}, e^{-i\alpha t})$, & non-singular o/w
The Generalized Graph Laplacian

- Parametric \( L_\alpha = d_\alpha I_N - A \), with \( d_\alpha = \sum_{j=1}^{M} 2d_j \cos(\alpha j) \), \( \alpha \in \mathbb{C} \), and weights \( d_j = A_{i,(j+i)N} \), annihilates \( x = e^{\pm i\alpha t} \), \( t = [0 \ldots N - 1] \) on circulant graphs of bandwidth \( M \) [MSK, '17]

- On the simple cycle, the rows (columns) of \( L_{C,\alpha} \) have 2 exponential vanishing moments

- \( L_{C,\alpha} \) is singular for \( \alpha = 2\pi k/N, \ k \in \mathbb{N} \), with \( N(L_{C,\alpha}) = \text{span}(e^{i\alpha t}, e^{-i\alpha t}) \), & non-singular o/w

- **Lemma 1** For \( \alpha \neq 2\pi k/N, \ k \in \mathbb{N} \), \( L_{C,\alpha}^{-1} \) has entries

\[
L_{C,\alpha}^{-1}(m, n) = \frac{1}{(-e^{-i\alpha} + e^{i\alpha})(-1 + e^{i\alpha N})} e^{i\alpha |n-m|} + \frac{1}{(e^{-i\alpha} - e^{i\alpha})(-1 + e^{-i\alpha N})} e^{-i\alpha |n-m|},
\]

\( 0 \leq m, n \leq N - 1. \)

⇒ The rows (columns) of \( L_{C,\alpha}^{-1} \) are complex exponentials

- **Lemma 2** For \( \alpha = 2\pi k/N, \ k \in \mathbb{N} \) and \( \alpha \neq 0, k\pi \), \( L_{C,\alpha}^{\dagger} \) has entries

\[
L_{C,\alpha}^{\dagger}(m, n) = \frac{e^{i\alpha}}{2N} \left( \frac{2|n-m|(-1 + e^{2i\alpha}) + (N - 1) - e^{2i\alpha}(N + 1)}{(-1 + e^{2i\alpha})^2} \right) e^{i\alpha |n-m|} + \frac{e^{-i\alpha}}{2N} \left( \frac{2|n-m|(-1 + e^{-2i\alpha}) + (N - 1) - e^{-2i\alpha}(N + 1)}{(-1 + e^{-2i\alpha})^2} \right) e^{-i\alpha |n-m|}, \ 0 \leq m, n \leq N - 1.
\]

⇒ The rows (columns) of \( L_{C,\alpha}^{\dagger} \) are linear complex exponential polynomials
The Generalized Graph Laplacian

- On general circulant graphs $G_S$, we have $L_\alpha = L_{C, \alpha} P_\alpha$, where $P_\alpha$ is circulant of bandwidth $M - 1$ and depends on the graph connectivity.
- $P_\alpha$ is positive definite up to certain $\alpha \in \mathbb{C}$ and $G_S$, then $P_\alpha^{-1}$ invokes a localized perturbation.

**Thm. 3**

For $\alpha \neq 2\pi k/N, \ k \in \mathbb{N}$, the cosparse analysis and sparse synthesis models of $L_\alpha$ are equivalent, generating **perturbed complex exponentials**

$$P_\alpha^{-1} L_{C, \alpha}^{-1} \Psi_T^{\Lambda^c}, \ \Lambda^c \subset V.$$

For $\alpha = 2\pi k/N, \ \alpha \neq 0, k\pi, \ k \in \mathbb{N}$, the sparse synthesis model generates **perturbed linear complex exponential polynomials**

$$P_\alpha^{-1} L_{C, \alpha}^\dagger \Psi_T^{\Lambda^c}, \ \Lambda^c \subset V.$$

The cosparse analysis model generates the **constrained, translated** subspaces

$$N(L_\alpha) + P_\alpha^{-1} L_{C, \alpha}^\dagger \Psi_T^{\Lambda^c} W_\alpha c, \ \Lambda^c \subset V,$$

for constraint $W_\alpha \in \mathbb{C}^{\mid \Lambda^c \mid \times \mid \Lambda^c \mid - 2}$ such that $\Psi_T^{\Lambda^c} (W_\alpha)_j \perp e^{\pm i\alpha t}$.

If $\Psi_T^{\Lambda^c} W_\alpha c = (L_{C, \alpha})_j$ for some $j \in V$, this reduces to **perturbed complex exponentials**

$$N(L_\alpha) + P_\alpha^{-1} \left( I_N - \frac{1}{N} E_\alpha \right) \tilde{c}, \ \text{for suitable } \tilde{c} \in \mathbb{C}^N$$

where $E_\alpha$ is the projection onto $N(L_\alpha)$, and is **comparable in order** to the case $\alpha \neq 2\pi k/N, \ k \in \mathbb{N}$.

- For $\alpha = 0$, this reduces to the graph Laplacian $L$
Figure 2: Comparison of signal models on circulant graphs for $L^{-1}_{\alpha}$, $\alpha = 0.21$.

Figure 3: Comparison of signal models on circulant graphs for $L^{\dagger}_{\alpha}$, $\alpha = 4\pi / N$. 
Uniqueness Guarantees

- Suppose $x$ belongs to a graph Laplacian based UoS model on an undirected graph:
  - Identify the unique (co)sparse solution of $y = Mx$, for suitable $M \in \mathbb{R}^{m \times N}$, $m < N$, with linearly independent rows

- Given mutually independent $M$ and $\Omega$, we require $m \geq \tilde{\kappa}_\Omega(l)$, with
  
  $$\tilde{\kappa}_\Omega(l) := \max\{\dim(W_{\Lambda_1} + W_{\Lambda_2}) : |\Lambda_i| \geq l, i = 1, 2\}$$

  to uniquely identify $x$ with $\Omega_{\Lambda}x = 0_{\Lambda}$, $l = N - ||\Omega x||_0$ [Lu et al, '07]
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  to uniquely identify \( x \) with \( \Omega_\Lambda x = 0_\Lambda, \ l = N - ||\Omega x||_0 \) [Lu et al, '07]

- **Corollary**: For mutually independent \( M \in \mathbb{R}^{m \times N} \) and \( \Omega = L \) on \( G = (V, E) \), the problem
  
  \[ Mx = y \text{ with } ||Lx||_0 \leq N - l = k \]

  has at most one solution, provided \( k > 1 \), if

  (i) \( m \geq 2k - 1 \), when the graph is connected,

  (ii1) \( m \geq 2k - 2 + c \), when the graph is disconnected with \( c \) components.

  (ii2) If \( x \in \bigcup_{|\Lambda| = l} W_\Lambda \), for \( W_\Lambda := N(L_\Lambda) \), subject to \( |\Lambda_i| < N_i - 1 \), (ii1) becomes \( m \geq 2k - c \).

- For a stable sampling scheme, \( m \) necessarily depends on \( \ln(L) \) and \( K \), for \( L \) total subspaces with maximum dimension \( K \) in a UoS [Blumensath et al, '09]

  \( \Rightarrow \) Model-based Compressed Sensing (on Graphs)
Conclusion and Future Work

- We have substantiated the discrepancy between the cosparse analysis and sparse synthesis models for the graph Laplacian through subspace analysis.
- We have characterized the functions defining the respective model subspaces on circulant graphs.
- For the parametric graph Laplacian on circulant graphs, we have shown transitional properties between model equivalence and non-equivalence.

For a comprehensive discussion, refer to arXiv Analysis vs Synthesis - An Investigation of (Co)sparse Signal Models on Graphs https://arxiv.org/abs/1811.04493

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Conclusion and Future Work

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- We have characterized the functions defining the respective model subspaces on circulant graphs.

- For the parametric graph Laplacian on circulant graphs, we have shown transitional properties between model equivalence and non-equivalence.

⇒ Develop refined UoS signal models on graphs with enhanced sampling schemes & recovery guarantees.

For a comprehensive discussion, refer to arXiv

*Analysis vs Synthesis with Structure - An Investigation of Union of Subspace Models on Graphs*

https://arxiv.org/abs/1811.04493
Thank you.
References I


