Portfolio Cuts

A Graph-Theoretic Framework to Diversification

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Mean-variance optimization (MVO)

Modern portfolio theory (introduced by Harry Markowitz in 1952) suggests an optimal investment strategy based on the first- and second-order moments of the asset returns $\Rightarrow$ **mean-variance optimization** (MVO).

Consider the vector, $\mathbf{r}(t) \in \mathbb{R}^N$, which contains the returns of N assets at a time t, the i-th entry of which is given by

$$r_i(t) = \frac{p_i(t) - p_i(t-1)}{p_i(t-1)} \tag{1}$$

where $p_i(t)$ denotes the value of the i-th asset at a time t.

The MVO asserts that the optimal vector of asset holdings, $\mathbf{w} \in \mathbb{R}^N$, is obtained from

$$\max_{\mathbf{w}} \left\{ \mathbf{w}^T \boldsymbol{\mu} - \lambda \mathbf{w}^T \boldsymbol{\Sigma} \mathbf{w} \right\} \tag{2}$$

where $\boldsymbol{\mu} = E\{\mathbf{r}\} \in \mathbb{R}^N$ is a vector of expected future returns, $\boldsymbol{\Sigma} = \mathrm{cov}\{\mathbf{r}\} \in \mathbb{R}^{N \times N}$ is the covariance matrix of returns, and λ is a Lagrange multiplier.

Issues with mean-variance optimization (MVO) I

There are a number of issues that make MVO unreliable in practice:

- Sensitivity of MVO to perturbations of the estimates of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$
  $\Rightarrow$ small changes in $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ may generate vstly different portfolio holdings $\mathbf{w}$
- The inputs, $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$, are time-varying
  - portfolios are never truly optimal since estimates are lagged
  - requires excessive turnover to adapt to changes
- The expected returns $\boldsymbol{\mu}$ can be rarely forecasted with sufficient accuracy

Issues with mean-variance optimization (MVO) II

Consequently, various risk-based asset allocation approaches have been proposed, which drop the term $\boldsymbol{\mu}$ altogether, with the optimization performed using $\boldsymbol{\Sigma}$ only.

The most important example is the **minimum variance** (MV) portfolio, formulated as

$$\min_{\mathbf{w}} \ \mathbf{w}^T \boldsymbol{\Sigma} \mathbf{w}, \quad \text{s.t.} \ \mathbf{w}^T \mathbf{1} = 1 \tag{3}$$

where $\mathbf{1} = [1, ..., 1]^T$.

The constraint, $\mathbf{w}^T \mathbf{1} = 1$, enforces full investment of the capital.

The optimal portfolio holdings then become

$$\mathbf{w} = \frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}^T \boldsymbol{\Sigma}^{-1}\mathbf{1}} \tag{4}$$

Instability issues remain prominent, as the matrix inversion of $\boldsymbol{\Sigma}$ required in (4) may lead to significant errors for ill-conditioned (singular) matrices.

Singularity of covariance matrices in practice I

The numerical instability issues associated with MV portfolio optimisation leads to a counter-intuitive result:

- The more collinear the asset returns
  $\Rightarrow$ the more unstable the portfolio solution (inversion of matrices in (4))
  $\Rightarrow$ the greater the need for diversification
- Increasing the size of $\mathbf{\Sigma}$ (more assets) further complicates the problem as more data samples are required to yield a positive-definite estimate $\Rightarrow$ at least $\frac{1}{2}(N^2 + N)$ independent and identically distributed (*i.i.d.*) observations of $\mathbf{r}(t)$ are needed

The severe impact of these challenges is highlighted by the fact that, in practice, even naive (equally-weighted) portfolios, i.e. $\mathbf{w} = \frac{1}{N}\mathbf{1}$, have been shown to outperform the mean-variance and risk-based optimization solutions.

How do we employ the covariance information without encountering these issues? $\Rightarrow$ **market graph models**

Market graph models I

A universe of assets can be modelled as a network of vertices on a **graph**, whereby an edge between two vertices (assets) designates both the existence of a link and the degree of similarity between assets.

A graph, $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$, is defined as a set of N vertices, $\mathcal{V} = \{1, 2, ..., N\}$, which are connected by a set of edges, $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$. The existence of an edge between vertices m and n is designated by $(m, n) \in \mathcal{E}$.

The strength of graph connectivity of an N-vertex graph can be represented by the *weight matrix*, $\mathbf{W} \in \mathbb{R}^{N \times N}$, with the entries defined as

$$W_{mn} \begin{cases} > 0, & (m, n) \in \mathcal{E}, \\ = 0, & (m, n) \notin \mathcal{E}, \end{cases} \tag{5}$$

thus conveying information about the *relative* importance of the vertex (asset) connections.

Market graph models II

The *degree matrix*, $\mathbf{D} \in \mathbb{R}^{N \times N}$, is a diagonal matrix with elements defined as

$$D_{mm} = \sum_{n=1}^{N} W_{mn} \tag{6}$$

and, and such, it quantifies the *centrality* of each vertex in a graph. Another important descriptor of graph connectivity is the graph *Laplacian matrix*, $\mathbf{L} \in \mathbb{R}^{N \times N}$, defined as

$$\mathbf{L} = \mathbf{D} - \mathbf{W} \tag{7}$$

which serves as an operator for evaluating the curvature, or smoothness, of the graph topology.

Market graph models III

A universe of N assets can be represented as a set of vertices on a *market graph*, whereby the edge weight, W_{mn}, between vertices m and n is defined as the absolute correlation coefficient, $|\rho_{mn}|$, of their respective returns of assets m and n, that is

$$W_{mn} = \frac{|\sigma_{mn}|}{\sqrt{\sigma_{mm}\sigma_{nn}}} = |\rho_{mn}| \tag{8}$$

where $\sigma_{mn} = \text{cov}\{r_m(t), r_n(t)\}$ is the covariance of returns between the assets m and n.

In this way, we have $W_{mn} = 0$ if the assets m and n are statistically independent (not connected), and $W_{mn} > 0$ if they are statistically dependent (connected on a graph).

Market graph models IV

Investment returns naturally reside on **irregular** (directed and sparse) graph domains.

Referring back to the MV in (4), recall that the covariance matrix $\boldsymbol{\Sigma}$ is **dense** in general $\Rightarrow$ standard multivariate models implicitly assume full connectivity of the graph, and are therefore **not adequate** to account for the structure inherent to real-world markets

Implicit graph-theoretic assumptions made by standard multivariate models:

- Undirected graph edges
- Completeness $\Rightarrow$ all graph vertices are connected to each other

It would be highly desirable to **remove unnecessary edges** in order to more appropriately model the underlying structure between assets (graph vertices) $\Rightarrow$ **portfolio cuts**

Portfolio cuts I

Consider an N-vertex market graph, $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$, which is grouped into $K = 2$ disjoint subsets of vertices, $\mathcal{V}_1 \subset \mathcal{V}$ and $\mathcal{V}_2 \subset \mathcal{V}$, with $\mathcal{V}_1 \cup \mathcal{V}_2 = \mathcal{V}$ and $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$.

A cut of this graph, for the given clusters, $\mathcal{V}_1$ and $\mathcal{V}_2$, is equal to a sum of all weights that correspond to the edges which connect the vertices between the subsets, $\mathcal{V}_1$ and $\mathcal{V}_2$, that is

$$Cut(\mathcal{V}_1, \mathcal{V}_2) = \sum_{m \in \mathcal{V}_1} \sum_{n \in \mathcal{V}_2} W_{mn} \tag{9}$$

A cut which exhibits the minimum value of the sum of weights between the disjoint subsets, $\mathcal{V}_1$ and $\mathcal{V}_2$, considering all possible divisions of the set of vertices, $\mathcal{V}$, is referred to as *the minimum cut*. Figure 1 provides an intuitive example of a graph cut.

Portfolio cuts II

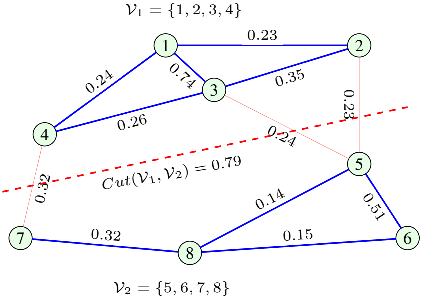


Figure: A cut for a graph with the disjoint subsets $\mathcal{V}_1 = \{1, 2, 3, 4\}$ and $\mathcal{V}_2 = \{5, 6, 7, 8\}$. The edges between the sets $\mathcal{V}_1$ and $\mathcal{V}_2$ are designated by thin red lines. The cut is equal to the sum of the weights that connect the sets $\mathcal{V}_1$ and $\mathcal{V}_2$, that is, $Cut(\mathcal{V}_1, \mathcal{V}_2) = 0.32 + 0.24 + 0.23 = 0.79$.

Portfolio cuts III

Within graph cuts, a number of optimization approaches may be employed to enforce some desired properties on graph clusters:

(i) *Normalized minimum cut*. The value of $Cut(\mathcal{V}_1, \mathcal{V}_2)$ is regularised by an additional term to enforce the subsets, $\mathcal{V}_1$ and $\mathcal{V}_2$, to be *simultaneously as large as possible*. The normalized cut formulation is given by

$$CutN(\mathcal{V}_1, \mathcal{V}_2) = \left(\frac{1}{N_1} + \frac{1}{N_2}\right) \sum_{m \in \mathcal{V}_1} \sum_{n \in \mathcal{V}_2} W_{mn} \tag{10}$$

where N_1 and N_2 are the respective numbers of vertices in the sets $\mathcal{V}_1$ and $\mathcal{V}_2$. Since $N_1 + N_2 = N$, the term $\frac{1}{N_1} + \frac{1}{N_2}$ reaches its minimum for $N_1 = N_2 = \frac{N}{2}$.

Portfolio cuts IV

(ii) *Volume normalized minimum cut*. Since the vertex weights are involved when designing the size of subsets $\mathcal{V}_1$ and $\mathcal{V}_2$, then by defining *the volumes* of these sets as $V_1 = \sum_{n \in \mathcal{V}_1} D_{nn}$ and $V_2 = \sum_{n \in \mathcal{V}_2} D_{nn}$, we arrive at

$$CutV(\mathcal{V}_1, \mathcal{V}_2) = \left(\frac{1}{V_1} + \frac{1}{V_2}\right) \sum_{m \in \mathcal{V}_1} \sum_{n \in \mathcal{V}_2} W_{mn} \tag{11}$$

Since $V_1 + V_2 = V$, the term $\frac{1}{V_1} + \frac{1}{V_2}$ reaches its minimum for $V_1 = V_2 = \frac{V}{2}$. Notice that vertices with a higher degree, D_{nn}, are considered as structurally more important than those with lower degrees. In turn, for market graphs, assets with a higher average statistical dependence to other assets are considered as more *central*.

Portfolio cuts V

Finding the minimum graph cut is an **NP-hard problem** $\Rightarrow$ to overcome the computational burden of finding the normalized minimum cut, we employ an **approximative spectral solution** which clusters vertices using the eigenvectors of **L**.

(i) *Normalized minimum cut*. It can be shown that if an **indicator vector** is defined as

$$x(n) = \begin{cases} \frac{1}{N_1}, & \text{for } n \in \mathcal{V}_1, \\ -\frac{1}{N_2}, & \text{for } n \in \mathcal{V}_2, \end{cases} \tag{12}$$

then the normalized cut, $CutN(\mathcal{V}_1, \mathcal{V}_2)$ in (10), is equal to the Rayleigh quotient of **L** and **x**, that is

$$CutN(\mathcal{V}_1, \mathcal{V}_2) = \frac{\mathbf{x}^T \mathbf{L} \mathbf{x}}{\mathbf{x}^T \mathbf{x}} \tag{13}$$

Therefore, the indicator vector, **x**, which minimizes the normalized cut also minimizes (13). This minimization problem, for the unit-norm form of the indicator vector, can also be written as

$$\min_{\mathbf{x}} \;\; \mathbf{x}^T \mathbf{L} \mathbf{x}, \quad \text{s.t.} \;\; \mathbf{x}^T \mathbf{x} = 1 \tag{14}$$

Portfolio cuts VI

which can be solved through the eigenanalysis of $\mathbf{L}$, that is

$$\mathbf{L}\mathbf{x} = \lambda_k \mathbf{x} \tag{15}$$

After neglecting the trivial solution $\mathbf{x} = \mathbf{u}_1$, $(k = 1)$, since it produces a constant eigenvector, we next arrive at $\mathbf{x} = \mathbf{u}_2$, $(k = 2)$.

Portfolio cuts VII

(ii) *Volume normalized minimum cut*. Similarly, by defining $\mathbf{x}$ as

$$x(n) = \begin{cases} \frac{1}{V_1}, & \text{for } n \in \mathcal{V}_1, \\ -\frac{1}{V_2}, & \text{for } n \in \mathcal{V}_2, \end{cases} \tag{16}$$

the volume normalized cut, $CutV(\mathcal{V}_1, \mathcal{V}_2)$ in (11), takes the form of a generalised Rayleigh quotient of $\mathbf{L}$, given by

$$CutV(\mathcal{V}_1, \mathcal{V}_2) = \frac{\mathbf{x}^T\mathbf{L}\mathbf{x}}{\mathbf{x}^T\mathbf{D}\mathbf{x}} \tag{17}$$

The minimization of (17) can be formulated as

$$\min_{\mathbf{x}} \ \mathbf{x}^T\mathbf{L}\mathbf{x}, \quad \text{s.t. } \mathbf{x}^T\mathbf{D}\mathbf{x} = 1 \tag{18}$$

which reduces to a generalized eigenvalue problem of $\mathbf{L}$, given by

$$\mathbf{L}\mathbf{x} = \lambda_k \mathbf{D}\mathbf{x} \tag{19}$$

Portfolio cuts VIII

Therefore, the solution to (18) becomes the generalized eigenvector of the graph Laplacian that corresponds to its lowest non-zero eigenvalue, that is $\mathbf{x} = \mathbf{u}_2$, $(k = 2)$.

For the spectral solutions above, the membership of a vertex, n, to either the subset $\mathcal{V}_1$ or $\mathcal{V}_2$ is uniquely defined by the *sign* of the indicator vector $\mathbf{x} = \mathbf{u}_2$, that is

$$\mathrm{sign}(x(n)) = \begin{cases} 1, & \text{for } n \in \mathcal{V}_1, \\ -1, & \text{for } n \in \mathcal{V}_2. \end{cases} \tag{20}$$

Notice that a scaling of $\mathbf{x}$ by any constant would not influence the solution for clustering into subsets $\mathcal{V}_1$ or $\mathcal{V}_2$.

Repeated portfolio cuts I

Although the above analysis has focused on the case with $K = 2$ disjoint sub-graphs, it can be straightforwardly generalized to $K \geq 2$ disjoint sub-graphs through the method of *repeated bisection*.

Figure 2 illustrates the hierarchical structure resulting from $K = 4$ portfolio cuts of a market graph, $\mathcal{G}$. The leaves of the resulting binary tree are given by $\{\mathcal{G}_3, \mathcal{G}_4, \mathcal{G}_5, \mathcal{G}_7, \mathcal{G}_8\}$ (in red), whereby the number of disjoint sub-graphs is equal to $(K + 1) = 5$. Notice that the union of the leaves equals to the original graph, i.e. $\mathcal{G}_3 \cup \mathcal{G}_4 \cup \mathcal{G}_5 \cup \mathcal{G}_7 \cup \mathcal{G}_8 = \mathcal{G}$.

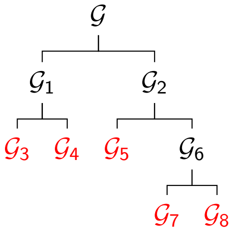


Figure: Graph structure resulting from $K = 4$ portfolio cuts

Graph asset allocation schemes I

The aim is to determine a diversified weighting scheme by distributing capital among the disjoint clusters (leaves) so that highly correlated assets within a given cluster receive the same total allocation, thereby being treated as a single uncorrelated entity.

By denoting the portion of the total capital allocated to a cluster $\mathcal{G}_i$ by w_i, we consider two simple asset allocation schemes:

(AS1) $w_i = \frac{1}{2^{K_i}}$, where K_i is the number of portfolio cuts required to obtain $\mathcal{G}_i$;

(AS2) $w_i = \frac{1}{K+1}$, where $(K+1)$ is the number of disjoint sub-graphs.

An equally-weighted asset allocation strategy may now be employed within each cluster, i.e. every asset within the i-th cluster, $\mathcal{G}_i$, will receive a weighting equal to $\frac{w_i}{N_i}$.

Graph asset allocation schemes II

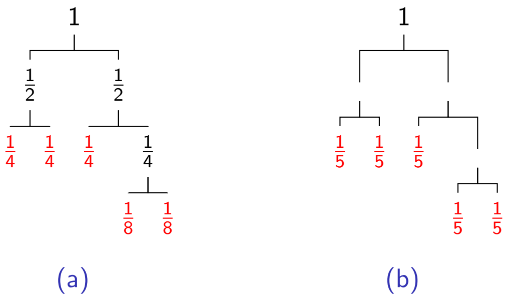


Figure: Graph asset allocation strategies. (a) $\frac{1}{2^{K_i}}$ scheme. (b) $\frac{1}{K+1}$ scheme.

Figures 3(a) and 3(b) demonstrate respectively the asset allocation schemes in AS1 and AS2 for $K = 4$ portfolio cuts, based on the market graph partitioning in Figure 2. Notice that the weights associated to the disjoint sub-graphs (leaves in red) sum up to unity.

Simulations on S&P 500 stocks I

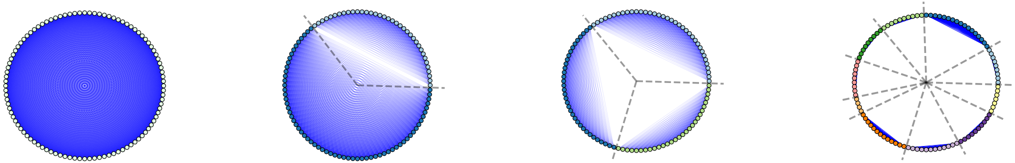
The performance of the portfolio cuts and the associated graph-theoretic asset allocation schemes was investigated using historical price data comprising of the 100 most liquid stocks in the S&P 500 index, based on average trading volume, in the period 2014-01-01 to 2018-01-01.

The data was split into:

1. *In-sample* dataset (2014-01-01 to 2015-12-31) which was used to estimate the asset correlation matrix and to compute the portfolio cuts
2. *Out-sample* (2016-01-01 to 2018-01-01), used to objectively quantify the profitability of the asset allocation strategies.

Figure 4 displays the K-th iterations of the proposed normalised portfolio cut in (13), for $K = 1, 2, 10$, applied to the original 100-vertex market graph obtain from the in-sample data set.

Simulations on S&P 500 stocks II



(a) (b) (c) (d)

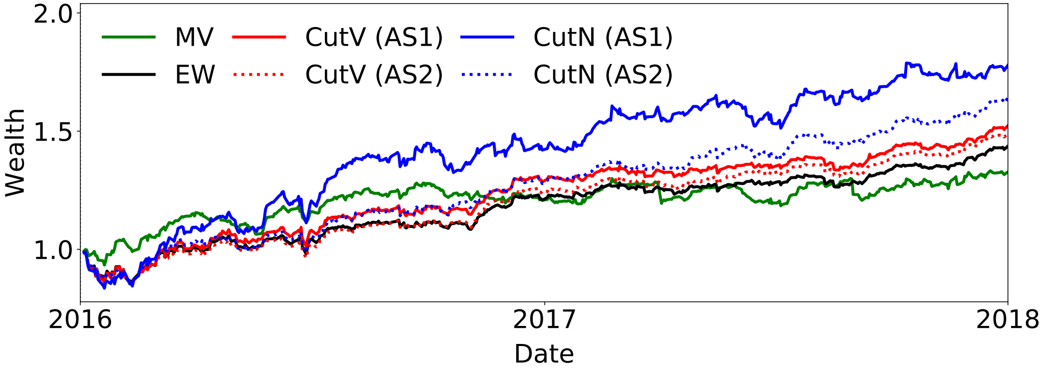
Figure: Visualisation of the 100-vertex market graph connectivity and its partitions into disjoint sub-graphs (separated by dashed grey lines). The edges (blue lines) were calculated based on the correlation between assets. (a) Fully connected market graph with 5050 edges. (b) Partitioned graph after $K = 1$ portfolio cuts (CutV), with 2746 edges. (c) Partitioned graph after $K = 2$ portfolio cuts (CutV), with 1731 edges. (d) Partitioned graph after $K = 10$ portfolio cuts (CutV), with 575 edges. Notice that the number of edges required to model the market graph is significantly reduced with each subsequent portfolio cut, since $\sum_{i=1}^{K+1} \frac{1}{2}(N_i^2 + N_i) < \frac{1}{2}(N^2 + N)$, $\forall K > 0$.

Simulations on S&P 500 stocks III

Next, for the out-sample dataset, graph representations of the portfolio, for the number of cuts K varying in the range $[1, 10]$, were employed to assess the performance of the graph asset allocation schemes. The standard equally-weighted (EW) and minimum-variance (MV) portfolios were also simulated for comparison purposes, with the results displayed in Figure 5.

The proposed graph asset allocations schemes consistently delivered lower out-sample variance than the standard EW and MV portfolios, thereby attaining a higher *Sharpe ratio*, i.e. the ratio of the mean to the standard deviation of portfolio returns. This verifies that the removal of possibly spurious statistical dependencies in the “raw” format, through the portfolio cuts, allows for robust and flexible portfolio constructions.

Simulations on S&P 500 stocks IV



(a) Evolution of wealth for both the traditional (EW and MV) and graph-theoretic asset allocation strategies, based on $(K = 10)$ portfolio cuts.

Simulations on S&P 500 stocks V

Cut Method	Allocation	$K = 1$	$K = 2$	$K = 3$	$K = 4$	$K = 5$	$K = 10$
CutV	AS1	1.82	1.80	1.80	1.93	1.96	**1.98**
CutV	AS2	1.82	1.81	1.94	2.03	1.95	**2.05**
CutN	AS1	1.93	2.01	2.08	2.23	2.22	**2.25**
CutN	AS2	1.93	2.04	2.17	**2.65**	2.51	2.48

(b) Sharpe ratios attained for varying number of portfolio cuts K.

Figure: Out-sample performance of the asset allocation strategies. Notice that the Sharpe ratio typically improves with each subsequent portfolio cut. The traditional portfolio strategies, EW and MV, attained the respective Sharpe ratios of $\mathrm{SR}_{\mathrm{EW}} = 1.85$ and $\mathrm{SR}_{\mathrm{MV}} = 1.6$.

Summary I

Portfolio cuts allow the investor to:

- Devise robust and tractable asset allocation schemes
- Consider smaller, computationally feasible, and economically meaningful clusters of assets, based on **graph cuts**
- fully utilize the asset returns covariance matrix for constructing the portfolio, even without the requirement for its inversion.