Fast and Robust ADMM for Blind Super-resolution

{ Yifan Ran, Wei Dai } Department of Electrical and Electronic Engineering

Imperial College London

Abstract & Main Contributions

Though the blind super-resolution problem is nonconvex in nature, recent advance shows the feasibility of a convex formulation which gives the unique recovery guarantee. However, the convexification procedure is coupled with a huge computational cost and is therefore of great interests to investigate fast algorithms. The main contributions of this work are in the following two aspects:

- A highly efficient solver is proposed by employing a novel preconditioning scheme and a column-wise update strategy. To our best knowledge, it is the first fast ADMM for the blind super-resolution problem.
- Simulations confirm the high efficiency and robustness of the proposed solver. Particularly, it is shown to be around 100 times faster than CVX.

Observation Model

Consider a set of point sources represented by a superposition of spikes $\xi(t) = \sum_j^J c_{j=1}\delta_{\tau_j}(t)$, The observation is a convolution between $\xi(t)$ and the Point Spread Function (PSF), as

$$\bar{y}(t) = \sum_j^J c_{j=1}\delta_{\tau_j}(t) * \bar{g}_j(t) = \sum_j^J c_j \bar{g}_j(t-\tau_j).$$

After sampling and transferring to the Fourier domain, we obtain the measurements as

$$y(n) = \sum_j c_j g_j(n) e^{-i2\pi(n-1)\tau_j}, \quad n = 1, \cdots, N.$$

To alleviate the underdeterminess of the above system, fixed supspace assumption is applied. That is, we assume the set of PSFs $\{\boldsymbol{g}_j\}_{j=1}^J$ lives in a fixed subspace spanned by the columns of a known matrix $\boldsymbol{B} := \begin{bmatrix} \boldsymbol{b}_1 & \cdots & \boldsymbol{b}_N \end{bmatrix}^{\mathrm{H}}$, and the measurements become

$$y(n) = \sum_j c_j e^{-i2\pi(n-1)\tau_j} \boldsymbol{b}_n^{\mathrm{H}} \boldsymbol{h}_j, \quad n = 1, \cdots, N.$$

To achieve the unique recovery, first define $\boldsymbol{a}(\tau_j) := \begin{bmatrix} e^{i2\pi(0)\tau_j} & \cdots & e^{i2\pi(N-1)\tau_j} \end{bmatrix}^{\mathrm{T}}$, the measurements can then be equivalently cast into a lifted form

$$\boldsymbol{y} = \mathcal{B}(\boldsymbol{X}_0),$$

where $\boldsymbol{X}_0 := \sum_j c_j \boldsymbol{h}_j \boldsymbol{a}(\tau_j)^{\mathrm{H}}$ and $\mathcal{B}(\boldsymbol{X}_0) := \{\boldsymbol{b}_n^{\mathrm{H}} \boldsymbol{X}_0 \boldsymbol{e}_n\}_{n=1}^N$.

Formulations

To search for a unique structured matrix, we adopt the so-called Atomic norm as the regularizer

$$\|\boldsymbol{X}\|_{\mathcal{A}} = \inf \left\{ \sum_{j=1}^J |c_j| : \boldsymbol{X} = \sum_{j=1}^J c_j \boldsymbol{h}_j \boldsymbol{a}(\tau_j)^{\mathrm{H}} \right\},$$

which admits an SDP characterization $\|\boldsymbol{X}\|_{\mathcal{A}} =$

$$\inf_{\boldsymbol{u},\boldsymbol{W}} \left\{ \frac{1}{2}\mathrm{Tr}\left(\frac{\mathcal{T}(\boldsymbol{u})}{N} + \boldsymbol{W}\right) : \begin{bmatrix} \mathcal{T}(\boldsymbol{u}) & \boldsymbol{X}^{\mathrm{H}} \\ \boldsymbol{X} & \boldsymbol{W} \end{bmatrix} \succeq 0 \right\}.$$

The blind super-resolution problem under noise can then be solved by

$$\begin{aligned} \underset{\boldsymbol{W},\boldsymbol{X},\boldsymbol{u}}{\text{minimize}} \quad & \frac{1}{2}\|\boldsymbol{y} - \mathcal{B}(\boldsymbol{X})\|_2^2 + \frac{\gamma}{2}(u_1 + \mathrm{Tr}(\boldsymbol{W})) \\ \text{subject to} \quad & \begin{bmatrix} \mathcal{T}(\boldsymbol{u}) & \boldsymbol{X}^{\mathrm{H}} \\ \boldsymbol{X} & \boldsymbol{W} \end{bmatrix} \succeq 0, \end{aligned}$$

where $u_1 \in \mathbb{R}$ is the first entry of $\boldsymbol{u}$. By the proposed conditioing scheme and applying the ADMM framework, we arrive at the following final formulation:

$$\begin{aligned} \underset{\boldsymbol{W},\boldsymbol{X},\boldsymbol{u},\boldsymbol{Z}}{\text{minimize}} \quad & \frac{1}{2}\|\boldsymbol{y} - \mathcal{B}(\boldsymbol{X})\|_2^2 + \frac{\gamma}{2}(u_1 + \mathrm{Tr}(\boldsymbol{W})) \\ \text{subject to} \quad & \boldsymbol{Z} = \begin{bmatrix} \frac{1}{\alpha}\mathcal{T}(\boldsymbol{u}) & \boldsymbol{X}^{\mathrm{H}} \\ \boldsymbol{X} & \alpha\boldsymbol{W} \end{bmatrix} \\ & \boldsymbol{Z} \succeq 0. \end{aligned}$$

Efficiency & Robustness

Algorithm 1 ADMM-based Solver

Input: measurements $\boldsymbol{y}$, dictionary $\boldsymbol{B}$

1: Initialize: $\boldsymbol{\Lambda}^0 \leftarrow \boldsymbol{0}$, $\boldsymbol{Z}^0 \leftarrow \boldsymbol{0}$

2: **while** the stopping criteria not met **do**

3: **for** $j = 1, \cdots, N$ **do**

$$\boldsymbol{x}_j^{k+1} \leftarrow \left(\boldsymbol{I}_K - \frac{\boldsymbol{b}_j\boldsymbol{b}_j^{\mathrm{H}}}{2\rho + \boldsymbol{b}_j^{\mathrm{H}}\boldsymbol{b}_j}\right)\left(\frac{y_j\boldsymbol{b}_j}{2\rho} + \frac{\boldsymbol{\lambda}_{0,j}^k}{\rho} + \boldsymbol{z}_{0,j}^k\right)$$

end for

4: $\boldsymbol{u}^{k+1} \leftarrow \alpha\boldsymbol{D}\left(\mathcal{T}^*\left(\boldsymbol{Z}_{11}^k + \boldsymbol{\Lambda}_{11}^k/\rho\right) - \frac{\gamma\alpha}{2\rho}\boldsymbol{e}_1\right)$

5: $\boldsymbol{W}^{k+1} \leftarrow \frac{1}{\alpha}\left(\boldsymbol{Z}_{22}^k + \frac{1}{\rho}(\boldsymbol{\Lambda}_{22}^k - \frac{\gamma}{2\alpha}\boldsymbol{I}_K)\right)$

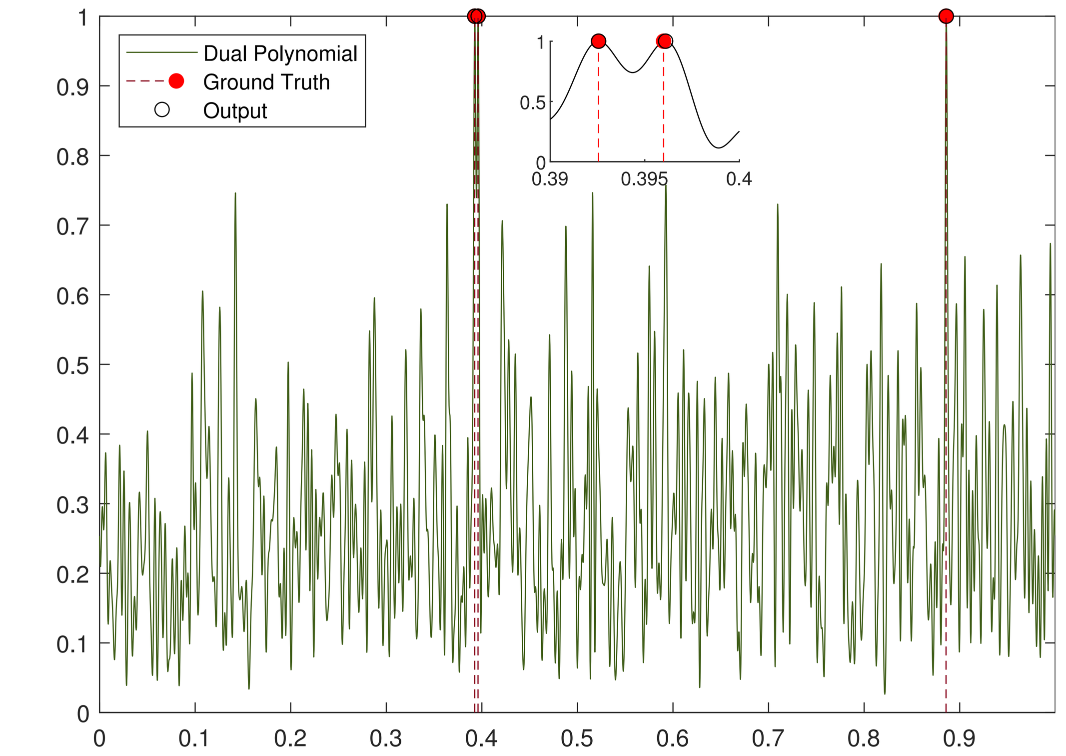
6: $\boldsymbol{Z}^{k+1} \leftarrow \Pi_{\mathbb{S}_+}\left(\begin{bmatrix} \frac{1}{\alpha}\mathcal{T}(\boldsymbol{u}) & \boldsymbol{X}^{\mathrm{H}} \\ \boldsymbol{X} & \alpha\boldsymbol{W} \end{bmatrix}^{k+1} - \boldsymbol{\Lambda}^k/\rho\right)$

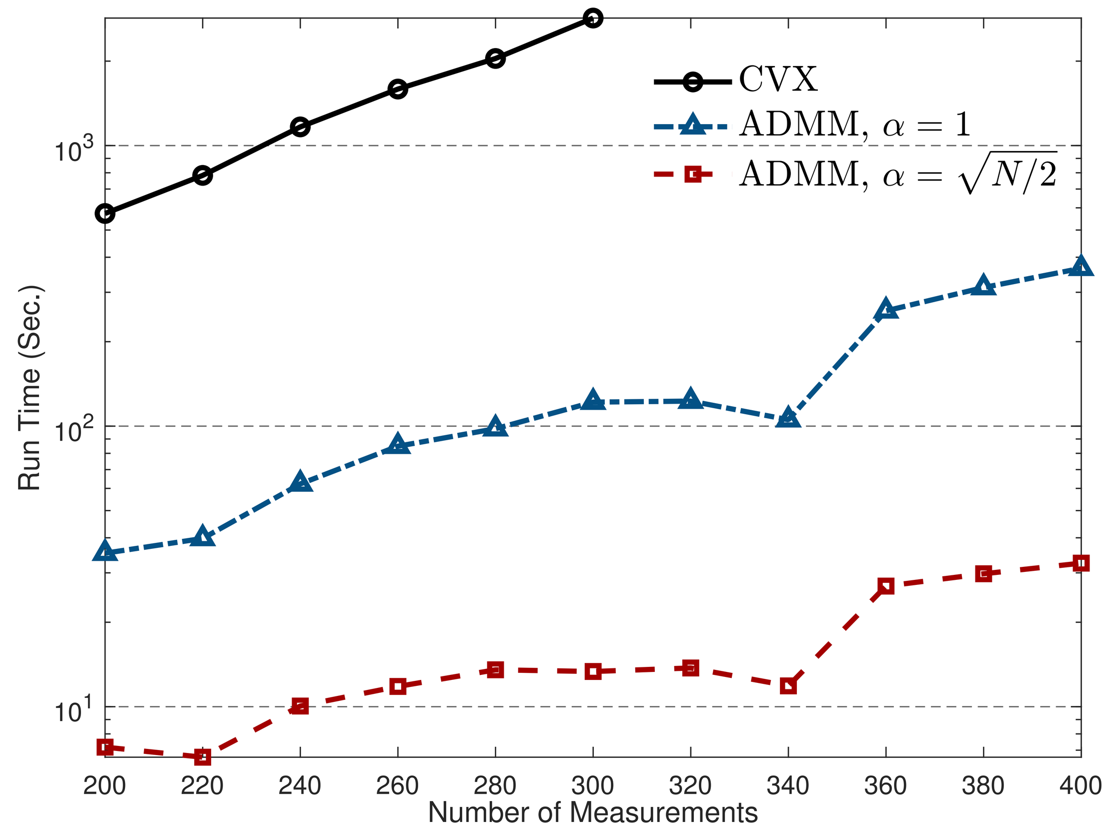
7: $\boldsymbol{\Lambda}^{k+1} \leftarrow \bar{\rho}\left(\boldsymbol{Z}^{k+1} - \begin{bmatrix} \frac{1}{\alpha}\mathcal{T}(\boldsymbol{u}) & \boldsymbol{X}^{\mathrm{H}} \\ \boldsymbol{X} & \alpha\boldsymbol{W} \end{bmatrix}^{k+1}\right) + \boldsymbol{\Lambda}^k$

8: **end while**

Stopping criteria:

9: $\left|\left\langle \begin{bmatrix} \mathcal{T}(\boldsymbol{u}) & \boldsymbol{X}^{\mathrm{H}} \\ \boldsymbol{X} & \boldsymbol{W} \end{bmatrix}^{k+1}, \begin{bmatrix} \frac{\gamma}{2N}\boldsymbol{I}_N & \boldsymbol{\Lambda}_0^{\mathrm{H}} \\ \boldsymbol{\Lambda}_0 & \frac{\gamma}{2}\boldsymbol{I}_K \end{bmatrix}^{k+1} \right\rangle\right| \leq \epsilon$





Conditioning & Stopping criteria

Theorem 1 (PSD cone conditioning) *For any positive real $\alpha \in \mathbb{R}_{++}$, the optimal points of the following program is invariant to the choice of α*

$$\begin{aligned} \textit{minimize} \quad & f_1(\boldsymbol{A}) + f_2(\boldsymbol{B}) + f_3(\boldsymbol{C}) \\ \textit{subject to} \quad & \begin{bmatrix} \frac{1}{\alpha}\boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{B}^H & \alpha\boldsymbol{C} \end{bmatrix} \succeq 0, \end{aligned}$$

where the objective functions are closed, proper and convex.

Proposition 1 (duality gap characterization) *The primal and dual solution pair $(\boldsymbol{X}, \boldsymbol{u}, \boldsymbol{W}, \boldsymbol{\Lambda})$ are optimal if and only if the following equation holds*

$$\left\langle \begin{bmatrix} \mathcal{T}(\boldsymbol{u}) & \boldsymbol{X}^H \\ \boldsymbol{X} & \boldsymbol{W} \end{bmatrix}, \begin{bmatrix} \frac{\gamma}{2N}\boldsymbol{I}_N & \boldsymbol{\Lambda}_0^H \\ \boldsymbol{\Lambda}_0 & \frac{\gamma}{2}\boldsymbol{I}_K \end{bmatrix} \right\rangle = 0,$$

where $\boldsymbol{I}_N$, $\boldsymbol{I}_K$ denotes the identity matrix of dimension $N \times N$ and $K \times K$, respectively.

Conclusion

In this work, we have proposed an ADMM-based convex solver for the blind suer-resolution problem. Its high efficiency and robustness are supported by the simulation results. Several new criteria and formulations are proposed.

Future Research

The proposed preconditioning scheme is expected to have a much broader impact. We are currently working on the theoretical charactrization of the proposed conditioning scheme including the optimal scaling factor and the corresponding improved rate.