Sequential structured dictionary learning for block sparse representations

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Introduction

• Dictionary learning methods have been applied to a number of signal and image processing applications.
• In some applications, the observed signal may have a multi-subspace structure that can be well exploited under the block-sparse signal representation framework.
• Using the observation that observed signals can be approximated as a sum of low rank matrices, we propose a new algorithm for learning a block-structured dictionary for block-sparse signal representations.
• Proposed algorithm not only performs better, but is also more efficient.
• This work is an extension of [2] and [3].

Background

Consider a data matrix $Y = \{y_1, y_2, \ldots, y_N\} \in \mathbb{R}^{n \times N}$ and a sparsity constraint $s$. DL algorithms seek to find a dictionary $D \in \mathbb{R}^{n \times K}$ and a sparse representation matrix $X \in \mathbb{R}^{K \times N}$ by optimizing the following cost function

$$\min_{D, X} \|Y - DX\|_F^2 \text{ s.t. } \|x_i\|_0 \leq s, \forall 1 \leq i \leq N,$$

$$\text{and } \|d_m\|_2 = 1, \forall 1 \leq m \leq K.$$ (1)

Optimization of the above cost function is carried out via alternating optimization of $D$ and $X$.

The proposed algorithm is based on a variant of (2) where the $\ell_2$-norm is used instead of the $\ell_1$-norm to give

$$\min_{D, X} \sum_{i=1}^{N} \left\| y_i - \sum_{j=1}^{J} D_{ij} x_{ij} \right\|_2^2 + \lambda \sum_{j=1}^{J} \sqrt{p_j} \| x_{ij} \|_2$$

and $\|d_m\|_2 = 1 \forall 1 \leq m \leq K$. (4)

The Proposed Approach

Our method is based on rewriting the first term of (4) as $\|Y - DX\|_F^2$ and observing that the matrix $DX$ can be expressed as a sum of block sparse low rank matrices or $Y_{ij} D_{ij}$. With an additional block incoherence constraint, the optimization problem (4) becomes

$$\sum_{j=1}^{J} \min_{D_{ij}} \|E_j - D_{ij} X_{ij}\|_F^2 + \lambda \sum_{j=1}^{J} \sqrt{p_j} \| x_{ij} \|_2$$

s.t. $D_{ij} J_{ij} = I_{p_j}$

where $x_{ij}$ is the $i$th column of $X$, $E_j = Y - \sum_{j=1,j \neq j}^{J} D_{ij} X_{ij}$, and $I_{p_j}$ is a $p_j$ dimensional identity matrix.

Algorithm Overview

Algorithm 1: The proposed Sequential Structured Dictionary Learning Algorithm.

Input: $Y$, $D_{ij} = \{D_1, D_2, \ldots, D_J\}$, $J$, $\epsilon_s$, $\lambda$, $B$

1. Ortho-normalize blocks of $D_{ij}$ set $\epsilon_2 = 0.01$
2. while $\|D_j - D_{A-1}\|_F / \|D_{A-1}\|_F \geq \epsilon_2$
3. for $j = 1$ do
4. $E_j = Y - \sum_{j=1,j \neq j}^{J} D_{ij} X_{ij}$
5. while $\|D_j - D_{A-1}\|_F / \|D_{A-1}\|_F \geq \epsilon_2$
6. Sparse Coding: for $i = 1$ do
7. $x_{ij} = \left( 1 - \frac{\|Y_i\|_2}{\|D_{ij} X_{ij}\|_2} \right) D_{ij} J_{ij}^{-1} e_{ij}$
8. Dictionary Update: Compute the SVD of
9. $E_j X_{ij}^T = U \Sigma V^T$
10. Update $D_j = U \Sigma V^T$
11. Output: $D_j$, $X_j$

Result: $D$ and $X$.

Computational Efficiency on Synthetic Exp

Table 1: Mean normalized reconstruction error over 100 trials, for multiple signal to noise ratios and signal block-sparsity levels $s$. $A$ is the block-sparsity controlling parameter and $s$ is the final block-sparsity level of the representation.

Table 2: Run-time for a single trial in seconds.

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<th>SNR in dB</th>
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References


