A Fixed-Point Iteration for Steady-State Analysis of Water Distribution Networks

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Motivation and contribution

- Water flow problem in water distribution networks
  - Compute the water flow rates in all pipes and the water pressure at all nodes
  - Nonlinear system of equations
- Fundamental task in water distribution network design and operation
  - [Mala-Jetmarova et al. '17] [Fooladivanda-Taylor '18] [Singh-Kekatos '18]
  - and joint optimization of energy and water networks in smart cities
    - [Dall’Anese-Mancarella-Monti '17] [Zamzam et al. '18] [Li et al. '18]
- Traditional solvers: Hardy-Cross, Newton-Raphson, Linear Theory Method
- Recent fixed-point method [Zhang et al. '17]
  - Improved convergence over industry standard (EPANET), but no analysis
- Existence/uniqueness of solution and algorithm convergence have been recognized as crucial in the literature
  - [Boulos-Altman-Liou '93] [Todini '06]
- Recent developments in fixed-point methods for power flow analysis
  - $1-\phi$ [Bolognani-Zampieri '16] [Wang et al.'18]; $3-\phi$ [Bazrafshan-Gatsis '18], [Bernstein et al.'18]
- Uniqueness of solution in natural gas networks [Singh-Kekatos '18]
- This paper: A fixed-point method for the water flow problem
  - Local uniqueness of solution, convergence, and rate of convergence
Water distribution network model

- Directed graph \((\mathcal{N}, \mathcal{L})\)
- \(\mathcal{N} = \{0, \ldots, N\}\) is the set of \(N + 1\) nodes
  - Node 0 is a reservoir
  - Rest of nodes are generically demands
- \(\mathcal{L} = \{1, \ldots, L\}\) is the set of \(L\) links: Pipes

- Hydraulic head at node \(n\) (proxy for pressure): \(h_n\)
- Rate of water injection at node \(n\): \(s_n \geq 0\) for reservoir, \(s_n \leq 0\) for junctions
- Rate of water flow in pipe \(\ell\): \(q_\ell\)
- Head loss across pipe \(\ell\) (pressure drop due to friction): \(\bar{h}_\ell\)

Hazen-Williams eq.: \(\bar{h}_\ell := \bar{h}_\ell(q_\ell) = A_\ell|q_\ell|^{0.852} q_\ell\)

where \(A_\ell\) is a constant that depends on the pipe characteristics

- Vectors \(s = \{s_n\}_{n \in \mathcal{N}_+}\); \(h = \{h_n\}_{n \in \mathcal{N}_+}\); \(s_\mathcal{N} = [s_0, s']'\); \(h_\mathcal{N} = [h_0, h']'\);
  \(q = \{q_\ell\}_{\ell \in \mathcal{L}}\); \(h = \{h_\ell\}_{\ell \in \mathcal{L}}\); \(h(q) = \{h_\ell(q_\ell)\}_{\ell \in \mathcal{L}}\)
Continuity and energy equations

- Graph incidence matrix $\mathcal{I}_N \in \mathbb{R}^{N+1} \times \mathbb{R}^L$

  $[\mathcal{I}_N]_{i,\ell} = \begin{cases} +1, & \text{if } \ell \text{ is directed out of node } i \\ -1, & \text{if } \ell \text{ is directed into node } i \end{cases}$

- **Continuity equation:** Rate of water injection into node $n \in \mathcal{N}$ equals the total rate of water flowing out on the links connected to node $n$

  $$s_N = \mathcal{I}_N q$$  \hspace{1cm} \text{(CE)}

- **Energy equation:** Head at the upstream node is equal to the head at the downstream node plus head losses occurring on the way

  $$\bar{h}(q) = \mathcal{I}'_N h_N$$  \hspace{1cm} \text{(EE)}
The Water Flow Problem

- Reservoir maintains constant head $h_0$
- Partition $\mathcal{I}_N = \begin{bmatrix} \mathcal{I}_0' \\ \mathcal{I} \end{bmatrix}$
  - $\mathcal{I}_0'$: Row corresponding to reservoir (node 0)
- The continuity and energy equations yield the Water Flow Equations:
  \begin{align*}
  s &= \mathcal{I} q, \quad (\text{WFE-1}) \\
  \bar{h}(q) &= \mathcal{I}' h - \mathcal{I}' 1_N h_0 \quad (\text{WFE-2})
  \end{align*}

- **Water Flow Problem**: Given the reservoir head $h_0$ and the injections $s$, determine the flow rates on all links, $q \in \mathbb{R}^L$, and the total head at all remaining nodes, $h \in \mathbb{R}^N$
- (WFE) is a system of $L + N$ nonlinear equations
**Fixed-point map: Derivation (1)**

- Suppose that all flows are bounded away from zero
- Notation: Diagonal matrix \( A = \text{diag}(A_1, \ldots, A_L) \)
- \( \text{diag}(|q|^{-0.852}) \) with entries \( |q_\ell|^{-0.852} \) on the diagonal
- The Hazen-Williams eq. \( h_\ell = A_\ell |q_\ell|^{0.852} q_\ell \) is written as
  \[
  q = A^{-1} \text{diag}(|q|^{-0.852}) h
  \]
- Introducing the latter in the WFE we obtain
  \[
  s = \mathcal{I} q \\
  h(q) = \mathcal{I}' h - \mathcal{I}' 1_N h_0
  \]

**Lemma**

*In a connected graph with nonzero flow rates, \( \mathcal{I} A^{-1} \text{diag}(|q|^{-0.852}) \mathcal{I}' \) is invertible.*

Proof: The matrix is the weighted Laplacian of the graph and is pos. semidefinite
Fixed-point map: Derivation (2)

- It follows from the previous lemma that

\[ h - 1_N h_0 = [ \mathcal{I} A^{-1} \text{diag}(|q|^{-0.852}) \mathcal{I}']^{-1} s \]

- Multiplying with \( \mathcal{I}' \) and invoking WFE-2 yields

\[ h = \mathcal{I}' [ \mathcal{I} A^{-1} \text{diag}(|q|^{-0.852}) \mathcal{I}']^{-1} s \]

- Introducing the latter into the Hazen-Williams equation finally yields a \textit{fixed-point map} for the water flows \( q \):

\[ q = T_s(q) \]

where \( T_s(.) \) is parametrized by the injection vector \( s \):

\[ T_s(q) = A^{-1} \text{diag}(|q|^{-0.852}) \mathcal{I}' [ \mathcal{I} A^{-1} \text{diag}(|q|^{-0.852}) \mathcal{I}']^{-1} s \]
Convergence

- Any flow vector \( q \) that solves the water flow problem satisfies \( q = T_s(q) \) and vice versa
- Iterative method indexed by \( k = 1, 2, \ldots \) initialized with \( q^0 \)

\[
q^{k+1} = T_s(q^k)
\]

Proposition

- Suppose that \( q^* \) is a fixed-point of the map \( T_s(q) \), that is, \( q^* = T_s(q^*) \)
- Let \( J_s^* = \frac{\partial T_s(q)}{\partial q} \bigg|_{q=q^*} \) be the Jacobian of the map \( T_s(q) \) evaluated at \( q^* \)
- Let \( \rho(J_s^*) \) be the spectral radius of \( J_s^* \)

\( \Rightarrow \) If \( \rho(J_s^*) < 1 \), then \( T_s(q) \) is locally a contraction map around \( q^* \), and \( q^* \) is a locally unique fixed point

Proof: Based on Ostrowski Theorem [Ortega-Rheinboldt ’70]
If all eigenvalues of $J^*_s$ have magnitude less than one, and the method is initialized in a neighborhood of $q^*$, then convergence to $q^*$ is guaranteed. The solution is unique in this neighborhood. The proposition does not characterize the size of the neighborhood. The contraction property characterizes the speed of convergence.

- Distance between successive iterates decreases by a factor $\alpha \in (0, 1)$

$$\|q^{k+1} - q^k\|_\infty \leq \alpha\|q^k - q^{k-1}\|_\infty$$

- Distance decreases linearly when plotted on a log scale

$$\log\|q^{k+1} - q^k\|_\infty \leq k \log \alpha + \log\|q^1 - q^0\|_\infty$$

- $\alpha$ is roughly $\rho(J^*_s)$
Test network

- Simplified version of test network in EPANET User Manual
- Demands $s = [0, -150, -150, -200, -150, 0, -300]'$ gallons per minute; reservoir head $h_0 = 850$ feet
- $A_\ell = 4.727C_\ell^{-1.852}d_\ell^{-4.871}l_\ell$
- $d_\ell$ and $l_\ell$: diameter and length of circular pipe $\ell$ in feet
- $C_\ell$: Hazen-Williams roughness coefficient (unitless)

<table>
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<th>Pipe</th>
<th>Length (ft.)</th>
<th>Diam. (in.)</th>
<th>H-W $C$</th>
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<tr>
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<td>14</td>
<td>100</td>
</tr>
</tbody>
</table>
Numerical tests

- Convergence criterion:
  \[ \| q^k - T_s(q^k) \|_\infty \leq 0.1 \text{ GPM} \] (quite small)
- Convergence linear in the iteration index
- Solution very close to Matlab’s fsolve

- From the figure:
  \[ \frac{\| q^{k+1} - q^k \|_\infty}{\| q^k - q^{k-1} \|_\infty} \approx 0.85 \]
- Very close to \( \rho(J^*_s) = 0.8520 \)
Conclusions and future directions

- The water flow problem amounts to a nonlinear system in flows and heads
- A fixed-point method is developed when all links are pipes
- Jacobian of the map characterizes the convergence, at least locally

Future directions

- Comprehensive network model: Tanks and pumps
- Other (more accurate) head loss equations
- More sophisticated analysis of the fixed-point map
  - Conditions for global convergence
  - Uniqueness of solution in a larger region of the $q$-space