A Fixed-Point Iteration for Steady-State Analysis of Water Distribution Networks

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November 28, 2018

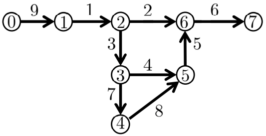
Acknowledgments: NSF Grants CMMI-1728629 and CCF-1421583

Motivation and contribution

- Water flow problem in water distribution networks
  - Compute the water flow rates in all pipes and the water pressure at all nodes
  - Nonlinear system of equations
- Fundamental task in water distribution network design and operation
  [Mala-Jetmarova *et al.* '17] [Fooladivanda-Taylor '18] [Singh-Kekatos '18]
  and joint optimization of energy and water networks in smart cities
  [Dall'Anese-Mancarella-Monti '17] [Zamzam *et al.* '18] [Li *et al.* '18]
- Traditional solvers: Hardy-Cross, Newton-Raphson, Linear Theory Method
- Recent fixed-point method [Zhang *et al.* '17]
  - Improved convergence over industry standard (EPANET), but no analysis
- Existence/uniqueness of solution and algorithm convergence have been recognized as crucial in the literature [Boulos-Altman-Liou '93] [Todini '06]
- Recent developments in fixed-point methods for power flow analysis
  - 1-ϕ [Bolognani-Zampieri '16] [Wang *et al.*'18]; 3-ϕ [Bazrafshan-Gatsis '18], [Bernstein *et al.*'18]
- Uniqueness of solution in natural gas networks [Singh-Kekatos '18]
- This paper: A fixed-point method for the water flow problem
  - Local uniqueness of solution, convergence, and rate of convergence

Water distribution network model

- Directed graph $(\mathcal{N}, \mathcal{L})$
- $\mathcal{N} = \{0, \ldots, N\}$ is the set of $N + 1$ nodes
  - Node 0 is a reservoir
  - Rest of nodes are generically demands
- $\mathcal{L} = \{1, \ldots, L\}$ is the set of L links: Pipes



- Hydraulic head at node n (proxy for pressure): h_n
- Rate of water injection at node n: $s_n \geq 0$ for reservoir, $s_n \leq 0$ for junctions
- Rate of water flow in pipe ℓ: q_ℓ
- Head loss across pipe ℓ (pressure drop due to friction): $\hbar_\ell$

$$\text{Hazen-Williams eq.:} \quad \hbar_\ell := \hbar_\ell(q_\ell) = A_\ell |q_\ell|^{0.852} q_\ell$$

where A_ℓ is a constant that depends on the pipe characteristics

- Vectors $s = \{s_n\}_{n \in \mathcal{N}_+}$; $h = \{h_n\}_{n \in \mathcal{N}_+}$; $s_\mathcal{N} = [s_0,\ s']'$; $h_\mathcal{N} = [h_0,\ h']'$; $q = \{q_\ell\}_{\ell \in \mathcal{L}}$; $\hbar = \{\hbar_\ell\}_{\ell \in \mathcal{L}}$; $\hbar(q) = \{\hbar_\ell(q_\ell)\}_{\ell \in \mathcal{L}}$

Continuity and energy equations

- Graph incidence matrix $\mathcal{I}_{\mathcal{N}} \in \mathbb{R}^{N+1} \times \mathbb{R}^{L}$

$$[\mathcal{I}_{\mathcal{N}}]_{i,\ell} = \begin{cases} +1, & \text{if } \ell \text{ is directed out of node } i \\ -1, & \text{if } \ell \text{ is directed into node } i \end{cases}$$

- *Continuity equation:* Rate of water injection into node $n \in \mathcal{N}$ equals the total rate of water flowing out on the links connected to node n

$$s_{\mathcal{N}} = \mathcal{I}_{\mathcal{N}} q \tag{CE}$$

- *Energy equation:* Head at the upstream node is equal to the head at the downstream node plus head losses occurring on the way

$$\hbar(q) = \mathcal{I}'_{\mathcal{N}} h_{\mathcal{N}} \tag{EE}$$

The Water Flow Problem

- Reservoir maintains constant head h_0
- Partition $\mathcal{I}_{\mathcal{N}} = \begin{bmatrix} \mathcal{I}_0' \\ \mathcal{I} \end{bmatrix}$
  - $\mathcal{I}_0'$: Row corresponding to reservoir (node 0)
- The continuity and energy equations yield the *Water Flow Equations:*

$$s = \mathcal{I}q, \tag{WFE-1}$$

$$\hbar(q) = \mathcal{I}'h - \mathcal{I}'\mathbf{1}_N h_0 \tag{WFE-2}$$

- *Water Flow Problem:* Given the reservoir head h_0 and the injections s, determine the flow rates on all links, $q \in \mathbb{R}^L$, and the total head at all remaining nodes, $h \in \mathbb{R}^N$
- (WFE) is a system of $L + N$ *nonlinear* equations

Fixed-point map: Derivation (1)

- Suppose that all flows are bounded away from zero
- Notation: Diagonal matrix $A = \mathrm{diag}(A_1, \ldots, A_L)$
- $\mathrm{diag}(|q|^{-0.852})$ with entries $|q_\ell|^{-0.852}$ on the diagonal
- The Hazen-Williams eq. $\hbar_\ell = A_\ell |q_\ell|^{0.852} q_\ell$ is written as

$$q = A^{-1}\mathrm{diag}(|q|^{-0.852})\hbar$$

- Introducing the latter in the WFE we obtain

$$\left.\begin{array}{r} s = \mathcal{I}q \\ \hbar(q) = \mathcal{I}'h - \mathcal{I}'\mathbf{1}_N h_0 \end{array}\right\} \Longrightarrow s = [\mathcal{I}A^{-1}\mathrm{diag}(|q|^{-0.852})\mathcal{I}'](h - \mathbf{1}_N h_0)$$

Lemma

In a connected graph with nonzero flow rates, $\mathcal{I}A^{-1}\mathrm{diag}(|q|^{-0.852})\mathcal{I}'$ is invertible.

Proof: The matrix is the weighted Laplacian of the graph and is pos. semidefinite

Fixed-point map: Derivation (2)

- It follows from the previous lemma that

$$h - \mathbf{1}_N h_0 = \left[\mathcal{I} A^{-1} \mathrm{diag}(|q|^{-0.852}) \mathcal{I}'\right]^{-1} s$$

- Multiplying with $\mathcal{I}'$ and invoking WFE-2 yields

$$\hbar = \mathcal{I}' \left[\mathcal{I} A^{-1} \mathrm{diag}(|q|^{-0.852}) \mathcal{I}'\right]^{-1} s$$

- Introducing the latter into the Hazen-Williams equation finally yields a *fixed-point map* for the water flows q:

$$q = T_s(q)$$

where $T_s(.)$ is parametrized by the injection vector s:

$$T_s(q) = A^{-1} \mathrm{diag}(|q|^{-0.852}) \mathcal{I}' \left[\mathcal{I} A^{-1} \mathrm{diag}(|q|^{-0.852}) \mathcal{I}'\right]^{-1} s$$

Convergence

- **Any flow vector q that solves the water flow problem satisfies $q = T_s(q)$ and vice versa**
- Iterative method indexed by $k = 1, 2, \ldots$ initialized with q^0

$$q^{k+1} = T_s(q^k)$$

Proposition

- *Suppose that q^* is a fixed-point of the map $T_s(q)$, that is, $q^* = T_s(q^*)$*
- *Let $J_s^* = \frac{\partial T_s(q)}{\partial q}|_{q=q^*}$ be the Jacobian of the map $T_s(q)$ evaluated at q^**
- *Let $\rho(J_s^*)$ be the spectral radius of J_s^**

➡ *If $\rho(J_s^*) < 1$, then $T_s(q)$ is locally a contraction map around q^*, and q^* is a locally unique fixed point*

Proof: Based on Ostrowski Theorem [Ortega-Rheinboldt '70]

Discussion

- If all eigenvalues of J_s^* have magnitude less than one, and the method is initialized in a neighborhood of q^*, then convergence to q^* is guaranteed
- The solution is unique in this neighborhood
- The proposition does not characterize the size of the neighborhood
- The contraction property characterizes the speed of convergence
  - Distance between successive iterates decreases by a factor $\alpha \in (0, 1)$

$$\|q^{k+1} - q^k\|_\infty \leq \alpha \|q^k - q^{k-1}\|_\infty$$

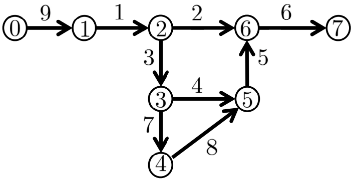
  - Distance decreases linearly when plotted on a log scale

$$\log \|q^{k+1} - q^k\|_\infty \leq k \log \alpha + \log \|q^1 - q^0\|_\infty$$

  - α is roughly $\rho(J_s^*)$

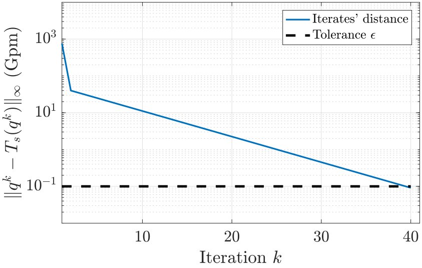
Test network

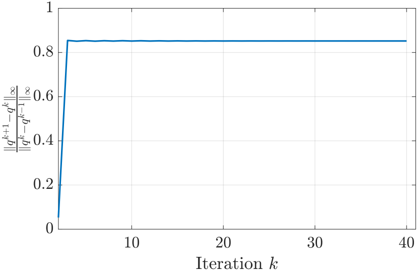
- Simplified version of test network in EPANET User Manual
- Demands $s = [0, -150, -150, -200, -150, 0, -300]'$ gallons per minute; reservoir head $h_0 = 850$ feet
- $A_\ell = 4.727 C_\ell^{-1.852} d_\ell^{-4.871} l_\ell$
- d_ℓ and l_ℓ: diameter and length of circular pipe ℓ in feet
- C_ℓ: Hazen-Williams roughness coefficient (unitless)



Pipe	Length (ft.)	Diam. (in.)	H-W C
1	3000	14	100
2	5000	12	100
3	5000	8	100
4	5000	8	100
5	5000	8	100
6	7000	10	100
7	5000	6	100
8	7000	6	100
9	3000	14	100

Numerical tests





- Convergence criterion: $\|q^k - T_s(q^k)\|_\infty \leq 0.1$ GPM (quite small)
- Convergence linear in the iteration index
- Solution very close to Matlab's `fsolve`

- From the figure: $\frac{\|q^{k+1}-q^k\|_\infty}{\|q^k-q^{k-1}\|_\infty} \approx 0.85$
- Very close to $\rho(J_s^*) = 0.8520$

Conclusions and future directions

- The water flow problem amounts to a nonlinear system in flows and heads
- A fixed-point method is developed when all links are pipes
- Jacobian of the map characterizes the convergence, at least locally

Future directions

- Comprehensive network model: Tanks and pumps
- Other (more accurate) head loss equations
- More sophisticated analysis of the fixed-point map
  - Conditions for global convergence
  - Uniqueness of solution in a larger region of the q-space

Citation: M. Bazrafshan, N. Gatsis, M. Giacomoni, and A. F. Taha, "A Fixed-Point Iteration for Steady-State Analysis of Water Distribution Networks," in *Proc. IEEE Global Conf. Signal and Information Processing*, Anaheim, CA, Nov. 2018. `https://arxiv.org/abs/1807.01404`