Multi-dimensional Signal Recovery using Low-rank Deconvolution

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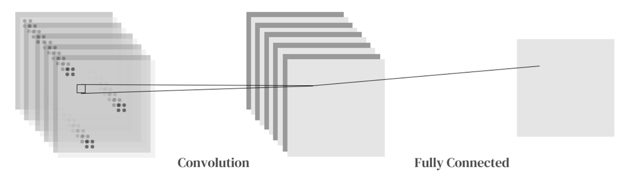
Background

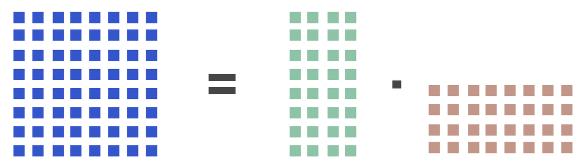
Convolution Sparse Coding

$$\mathbf{S} \approx \sum_m \mathbf{D}_m * \mathbf{X}_m$$



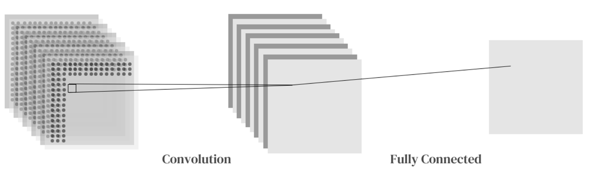
Low-rank Approximation

$$\mathbf{S} \approx \mathbf{X}^T \mathbf{Y}$$



Low-rank Deconvolution

$$\mathcal{S} \approx \sum_m \mathcal{D}_m * [\![\mathbf{X}_m^{(1)}, \ldots, \mathbf{X}_m^{(N)}]\!]$$



$$\mathbf{X}^{(n)} = [\mathbf{v}_1^{(n)}, \mathbf{v}_2^{(n)}, \ldots, \mathbf{v}_R^{(n)}]$$

$$\mathcal{J} = [\![\mathbf{X}^{(1)}, \mathbf{X}^{(2)}, \ldots, \mathbf{X}^{(N)}]\!] = \sum_{r=1}^{R} \mathbf{v}_r^{(1)} \circ \mathbf{v}_r^{(2)} \circ \ldots \circ \mathbf{v}_r^{(N)}$$

$$^{(n)}\mathbf{J} = \mathbf{X}^{(n)}(\mathbf{Q}^{(n)})^\top, \quad \mathbf{Q}^{(n)} = \mathbf{X}^{(N)} \odot \ldots \odot \mathbf{X}^{(n+1)} \odot \mathbf{X}^{(n-1)} \odot \ldots \odot \mathbf{X}^{(1)}$$

$$\hat{\mathbf{W}}_m^{(n)} = \hat{\mathbf{D}}_m^{(n)}[\hat{\mathbf{Q}}_m^{(n)} \otimes \mathbf{I}_{I_n}],$$
$$\hat{\mathbf{W}}^{(n)} = [\hat{\mathbf{W}}_0^{(n)}, \hat{\mathbf{W}}_1^{(n)}, \ldots, \hat{\mathbf{W}}_M^{(n)}],$$
$$\hat{\mathbf{x}}^{(n)} = [(\hat{\mathbf{x}}_0^{(n)})^\top, (\hat{\mathbf{x}}_1^{(n)})^\top, \ldots, (\hat{\mathbf{x}}_M^{(n)})^\top]^\top,$$
$$\hat{\mathbf{z}}^{(n)} = [(\hat{\mathbf{z}}_0^{(n)})^\top, (\hat{\mathbf{z}}_1^{(n)})^\top, \ldots, (\hat{\mathbf{z}}_M^{(n)})^\top]^\top,$$

Low-rank Deconvolution I

Problem Formulation

$$\underset{\{\mathbf{X}_m^{(n)}\}}{\arg\min} \frac{1}{2} \left\| \sum_{m=1}^{M} \mathcal{D}_m * [\![\mathbf{X}_m^{(1)}, \ldots, \mathbf{X}_m^{(N)}]\!] - \mathcal{S} \right\|_2^2 + \sum_{m=1}^{M} \sum_{n=1}^{N} \lambda \left\| \mathbf{X}_m^{(n)} \right\|_1$$

ADMM Formulation

$$\underset{\{\mathbf{X}_m^{(n)}\},\{\mathbf{Y}_m^{(n)}\}}{\arg\min} \frac{1}{2} \left\| \sum_{m=1}^{M} \mathcal{D}_m * [\![\ldots, \mathbf{X}_m^{(n)}, \ldots]\!] - \mathcal{S} \right\|_2^2 + \sum_{m=1}^{M} \lambda \left\| \mathbf{Y}_m^{(n)} \right\|_1$$

subject to $\mathbf{X}_m^{(n)} = \mathbf{Y}_m^{(n)} \; \forall m$

Formulation in the DFT domain

$$\underset{\{\hat{\mathbf{x}}_m^{(n)}\}}{\arg\min} \frac{1}{2} \left\| \sum_{m=1}^{M} \hat{\mathbf{D}}_m^{(n)} [\hat{\mathbf{Q}}_m^{(n)} \otimes \mathbf{I}_{I_n}] \hat{\mathbf{x}}_m^{(n)} - \hat{\mathbf{s}}^{(n)} \right\|_2^2 + \frac{\rho}{2} \sum_{m=1}^{M} \left\| \hat{\mathbf{x}}_m^{(n)} - \hat{\mathbf{z}}_m^{(n)} \right\|_2^2$$

$$\underset{\hat{\mathbf{x}}^{(n)}}{\arg\min} \frac{1}{2} \left\| \hat{\mathbf{W}}^{(n)} \hat{\mathbf{x}}^{(n)} - \hat{\mathbf{s}}^{(n)} \right\|_2^2 + \frac{\rho}{2} \left\| \hat{\mathbf{x}}^{(n)} - \hat{\mathbf{z}}^{(n)} \right\|_2^2$$

$$[(\hat{\mathbf{W}}^{(n)})^H \hat{\mathbf{W}}^{(n)} + \rho \mathbf{I}_\beta] \hat{\mathbf{x}}^{(n)} = (\hat{\mathbf{W}}^{(n)})^H \hat{\mathbf{s}}^{(n)} + \rho \hat{\mathbf{z}}^{(n)}$$

Low-rank Deconvolution II

Linear Mask Decoupling for Tensor Completion

$$\underset{\hat{\mathbf{x}}^{(n)}}{\arg\min} \frac{1}{2} \left\| \mathbf{P}^{(n)} \hat{\mathbf{F}}^{(n)} \hat{\mathbf{W}}^{(n)} \hat{\mathbf{x}}^{(n)} - \mathbf{s}^{(n)} \right\|_2^2 + \frac{\alpha}{2} \left\| \hat{\mathbf{x}}^{(n)} \right\|_2^2$$

$$[(\hat{\mathbf{T}}^{(n)})^H \hat{\mathbf{T}}^{(n)} + \alpha \mathbf{I}_\beta] \hat{\mathbf{x}}^{(n)} = (\hat{\mathbf{T}}^{(n)})^H \mathbf{s}^{(n)}$$

Mask matrix

$$\hat{\mathbf{T}}^{(n)} = \mathbf{P}^{(n)} \hat{\mathbf{F}}^{(n)} \hat{\mathbf{W}}^{(n)}$$

DFT matrix

Algorithms

while *not converged* do
- $\mathbf{X}_m^{(n)(k+1)} = \arg\min \frac{1}{2} \left\| \sum_{m=1}^{M} \mathcal{D}_m * [\![\ldots, \mathbf{X}_m^{(n)}, \ldots]\!] - \mathcal{S} \right\|_2^2 + \frac{\rho}{2} \sum_{m=1}^{M} \left\| \mathbf{X}_m^{(n)} - \mathbf{Y}_m^{(n)(k)} + \mathbf{U}_m^{(n)(k)} \right\|_2^2$
- $\mathbf{Y}_m^{(n)(k+1)} = \mathbf{prox}_{\mathbf{1}_{\frac{\lambda}{\rho}}} (\mathbf{X}_m^{(n)(k+1)} + \mathbf{U}_m^{(k)})$
- $\mathbf{U}_m^{(n)(k+1)} = \mathbf{U}_m^{(n)(k)} + \mathbf{X}_m^{(n)(k+1)} - \mathbf{Y}_m^{(n)(k+1)}$

end

Not.: $\mathbf{prox}_{\mathbf{1}_\gamma}(\mathbf{u}) = \text{sign}(\mathbf{u}) \oplus \max(0, |\mathbf{u}| - \gamma)$.

▲ **Algorithm 1:** ADMM algorithm for LRD. prox is proximal operator to perform shrinkage. sign(·), max(·) and norm of a vector considered to be applied element-wise.

input : $\mathcal{S}, \{\mathcal{D}_m\}_{m=1}^M, \{\mathbf{X}_{0,m}^{(n)}\}_{n=1,m=1}^{N,M}, R > 0$

output: $\{\mathbf{X}_m^{(n)}\}_{n=1,m=1}^{N,M}$

/* Initialize Kruskal Factors */

$\{\mathbf{X}_m^{(n)}\} = \{\mathbf{X}_{0,m}^{(n)}\}$

/* Main Loop, Eq. (8) */

while *not converged* do
- for $n = 1, \ldots, N$ do
  - $\mathbf{X}_m^{(n)} = \arg\min \frac{1}{2} \left\| \sum_{m=1}^{M} \mathcal{D}_m * [\![\mathbf{X}_m^{(1)}, \ldots, \mathbf{X}_m^{(N)}]\!] - \mathcal{S} \right\|_2^2 + \Phi(\{\mathbf{X}_m^{(n)}\})$
- end

end

▲ **Algorithm 2:** LRD algorithm. It solves the LRD problem by means of an alternaed approach for every n-mode.

Conclusion

- Low-rank Deconvolution is a sufficient prior to learn latent structure of data
  Compressed Video Reconstruction
- Low-rank Deconvolution is flexible enough to deal with incomplete data
  Image In-painting Problems
- Future work: higher data dimensions and larger datasets
  Importance of Compression

Results

Compressed Video Reconstruction

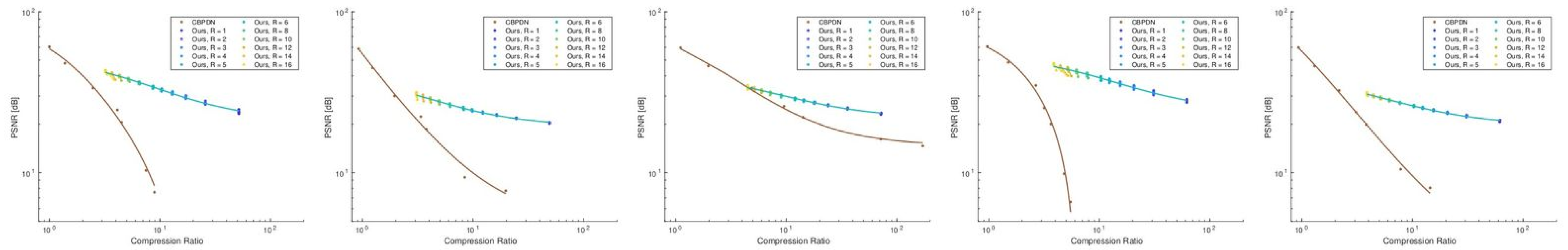


Image In-painting

Missing \ Image	Barbara	Boat	C.Man	Couple	F.Print	Hill	House	Man	Montage	Peppers	Av.
30%	26.80	23.64	26.96	24.29	20.20	25.86	30.28	22.17	27.74	23.13	25.11
50%	23.76	23.10	24.83	22.84	18.17	23.41	27.32	21.36	23.33	20.76	22.89
60%	22.48	22.54	24.35	22.25	17.60	22.93	25.52	20.62	22.76	20.39	22.14

▲ **TOP:** Qualitative and quantitative evaluation on image in-painting. **Top:** From left to right, we display ground truth, input and result for the Barbara image for a 50% missing pixels. **Bottom:** The table reports the PSNR in dB (higher is better) using our approach for 10 images We indicate the solution for a missing pixel rate of 30, 50 and 60%.

◀ **LEFT:** Compressed video reconstruction. **Top:** Quality of reconstruction (PSNR) vs. compression ratio (CR) for the sequences Basketball, Fotball1, Ironman, Skiing and Soccer respectively, evaluation on the CBDPN method and ours respectively. **Bottom:** Qualitative evaluation on RGB Basketball video for a similar CR~9. From top to bottom, ground truth, reconstruction using CBPDN method and our solution.

Paper