Sine-based EB-ESPRIT for source localization

2018. 07. 09.
Sheffield, United Kingdom

Byeongho Jo, Jung-Woo Choi
(byeongho@kaist.ac.kr, jwoo@kaist.ac.kr)

School of Electrical Engineering
Korea Advanced Institute of Science and Technology (KAIST)
South Korea
<table>
<thead>
<tr>
<th>Objective</th>
<th>Background</th>
<th>Problem Statement</th>
<th>Proposed method</th>
<th>Performance validation</th>
</tr>
</thead>
</table>

**Objective**

Direction-of-arrival estimation using spherical microphone array
Objective

- **Direction-of-Arrival (DOA) estimation**
  - Measurement pressure data $\rightarrow$ incoming directions of sources

- **Spherical microphone array**
  - Measure 3-D sound field
  - Processing in spherical harmonic domain
Objective

- **Subspace-based method**
  - Using orthogonality between signal and noise subspaces
  - ESPRIT: Parametric estimation (without grid-search)

- **EB-ESPRIT (Eigenbeam – Estimation of Signal Parameters via Rotational Invariance Techniques)**
  - Processing in spherical harmonic domain
  - Using a recurrence relation of spherical harmonics

- **Practical problem**
  - Singularity of tangent function
    (directional parameter of EB-ESPRIT)

Objective: solve the practical problem of EB-ESPRIT
Background

Signal processing in spherical harmonic domain
### Spherical Fourier transform (SFT)

The Spherical Fourier transform (SFT) is given by:

$$p_{nm} = \int 4\pi p(\Omega) Y_n^m(\Omega)^* d\Omega$$

The sound field from $D$ sources is:

$$p(\Omega) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \sum_{d=1}^{D} b_n(kr_0) Y_n^m(\Omega_s^{(d)})^* S^{(d)} Y_n^m(\Omega)$$

where $Y_n^m(\Omega)$ are spherical harmonics, $b_n(kr_0)$ is the radial dependency, and $S^{(d)}$ is the sound source.

For a sound source $s^{(d)}(\omega)$, spherical harmonics are given by:

$$Y_n^m(\theta, \phi) = \frac{2n+1}{4\pi} \frac{(n-m)!}{(n+m)!} P_n^m(\cos \theta) e^{im\phi}$$

where $P_n^m(\cos \theta)$ are Legendre polynomials.

Measured pressure data (spatial domain) is:

$$p_1(\omega) \ldots p_L(\omega)$$

Background | SFT of multiple plane waves

\[ p_\ell (\Omega) = \sum_{d=1}^{D} p^{(d)}_\ell (\Omega) \]

\[ a_{nm} = \sum_{d=1}^{D} Y^m_n (\Omega^{(d)}_s)^* S^{(d)} \]

Vector form: \( \mathbf{p} \)

\[ \mathbf{a} = \mathbf{Y}^H \mathbf{s} \]

Measured pressure data (spatial domain)

Directional harmonic coefficients (spherical harmonic domain)
Signal processing in spherical harmonic domain

1. Space domain

\[
p = \begin{bmatrix} p_1(\omega) \\ \vdots \\ p_L(\omega) \end{bmatrix}
\]

2. Spherical harmonic domain

\[
a = Y^H s = \begin{bmatrix} a_{0,0}(k) \\ a_{1,-1}(k) \\ \vdots \\ a_{N,N}(k) \end{bmatrix}
\]

3. Construction of covariance matrix

\[
R = E \left[ a a^H \right] 
\approx \frac{1}{J} \sum_{j=1}^{J} (a^{(j)} a^{(j)H})
\]

\(J\): number of observations

4. Eigenvalue Decomposition

\[
R = U \Sigma U^H 
= \begin{bmatrix} \hat{U} & U_n \end{bmatrix} \Sigma \begin{bmatrix} \hat{U} & U_n \end{bmatrix}^H
\]

\([D]\) largest eigenvalues

Corresponding eigenvectors
Signal processing in spherical harmonic domain

1. Space domain
\[ p = \begin{bmatrix} p_1(\omega) \\ \vdots \\ p_L(\omega) \end{bmatrix} \]

2. Spherical harmonic domain
\[ a = Y^H s = \begin{bmatrix} a_{0,0}(k) \\ a_{1,-1}(k) \\ \vdots \\ a_{N,N}(k) \end{bmatrix} \]

3. Construction of covariance matrix
\[ R = E\left[ aa^H \right] \approx \frac{1}{J} \sum_{j=1}^{J} (a^{(j)} a^{(j)H}) \]

4. Eigenvalue Decomposition
\[ R = U \Sigma U^H \]

Utilize the sub-matrix \( \hat{U} \) spanning signal subspace for EB-ESPRIT
Problem Statement

Singularity problem of EB-ESPRIT
Recurrence relation of spherical harmonics

\[ 2m Y_n^m (\Omega)^* + \Lambda^+ Y_{n+1}^{m+1} (\Omega)^* \tan \theta e^{i\phi} + \Lambda^- Y_{n-1}^{m-1} (\Omega)^* \tan \theta e^{-i\phi} = 0 \]

where \( \Lambda^\pm = \sqrt{n \mp m (n \pm m + 1)} \)

- Matrix form

\[ 2MY^H_{(0,0)} + \Lambda^+ Y^{(0,+1)}^H \Phi + \Lambda^- Y^{(0,-1)}^H \Phi^* = 0 \]

DOA information (unknown) ➔ What we want to know!

Relationship between signal subspaces and the directional coefficients

- Measurable data: \( a = Y^H s \)

\[ R = E[aa^H] = Y^H E[ss^H] Y = \hat{U} U_n \Sigma \hat{U} U_n^H \]

\[ \text{span}\{\hat{U}\} = \text{span}\{Y^H\} \]

\[ \hat{U} = Y^H T \quad T: \text{transformation matrix} \]
Recurrence relation for $Y^H$

\[ 2MY^{(0,0)H} + \Lambda^+ Y^{(0,+1)H} \Phi + \Lambda^- Y^{(0,-1)H} \Phi^* = 0 \]

Recurrence relation for $\hat{U}$

\[ 2M\hat{U}^{(0,0)} + \Lambda^+ \hat{U}^{(0,+1)} \Psi + \Lambda^- \hat{U}^{(0,-1)} \Psi^* = 0 \]

\[ \hat{U} = Y^H T \]

Solve the equation for $\Psi$

\[ \Psi = T^{-1} \Phi T \]

Eigen value decomposition of matrix $\Psi$

\[ \Phi \]

Relationship between $\Psi$ and $\Phi$

Matrix containing DOA information

\[ \hat{\Theta}_s^{(d)} = \tan^{-1} \left| \Phi_s^{(d)} \right|, \quad \hat{\phi}_s^{(d)} = \angle \Phi_s^{(d)} \]
EB–ESPRIT: DOA estimation in a parametric manner

Recurrence relation for $Y^{(0)}$

$$2MY^{(0,0)H} + \Lambda^+Y^{(0,+1)H}\Phi + \Lambda^-Y^{(0,-1)H}\Phi^* = 0$$

Recurrence relation for $\hat{U}$

$$2M\hat{U}^{(0,0)} + \Lambda^+\hat{U}^{(0,+1)}\Psi + \Lambda^-\hat{U}^{(0,-1)}\Psi^* = 0$$

$$\hat{U} = Y^HT$$

Relationship between $\Psi$ and $\Phi$

$$\Psi = T^{-1}\Phi T$$

Matrix containing DOA information

Eigen value decomposition of matrix $\Psi$

$$\Phi$$

EB–ESPRIT: DOA estimation in a parametric manner
Directional parameter for elevation: tangent function (singularity)

\[
2mY_n^m(\Omega_s)^* + \Lambda^+Y_n^{m+1}(\Omega_s)^* \tan \theta_s e^{i\phi_s} + \Lambda^-Y_n^{m-1}(\Omega_s)^* \tan \theta_s e^{-i\phi_s} = 0
\]

\[
\Phi_s \quad \Phi^*_s
\]

\[
\theta = \tan^{-1}(|\Phi_s|)
\]

Singularity problem

- Singularity of tangent function near \( \theta \approx \pi/2 \)
- Cannot estimate the DOAs when the sources near the equator

How do we overcome the singularity problem?
Prior works to overcome singularity problem

- [Sun et al, ICASSP 2011]
  - Rotation of the reference coordinate when the robustness measure is bad

- [Huang et al, IEEE T-ASLP 2017]
  - Two-stage decoupled approach (TSDA):
    - Unitary spherical ESPRIT (U-SHESPRIT) $\rightarrow \theta$ (cosine function)
    - Unitary spherical root-MUSIC (U-SHRMUSIC) $\rightarrow \phi$
      - Additional computations!
      - Find elevations and azimuth angles separately
      - 2nd order-reduction of spherical harmonic coefficients ($\theta$)
Prior works to overcome singularity problem

- [Sun et al, ICASSP 2011]
  - Rotation of the reference coordinate when the robustness measure is bad

- [Huang et al, IEEE T-ASLP 2017]
  - Two-stage decoupled approach (TSDA):
    - Unitary spherical ESPRIT (U-SHESPRIT) $\rightarrow$ $\theta$ (cosine function)
    - Unitary spherical root-MUSIC (U-SHRMUSIC) $\rightarrow$ $\phi$
  - Additional computations!
    - Find elevations and azimuth angles separately

How do we overcome the singularity problem without artifacts?
<table>
<thead>
<tr>
<th>Objective</th>
<th>Background</th>
<th>Problem Statement</th>
<th>Proposed method</th>
<th>Performance validation</th>
</tr>
</thead>
</table>

Proposed method

Sine-based EB-ESPRIT
Sine–based recurrence relations of spherical harmonics

\[ 2mY_n^m(\Omega_s)^* + w_{nm}^+ Y_{n-1}^{m+1}(\Omega_s)^* \sin \theta_s e^{i\phi_s} + w_{nm}^- Y_{n-1}^{m-1}(\Omega_s)^* \sin \theta_s e^{-i\phi_s} = 0 \]

\[ 2mY_{n-1}^m(\Omega_s)^* + v_{nm}^+ Y_n^{m+1}(\Omega_s)^* \sin \theta_s e^{i\phi_s} + v_{nm}^- Y_n^{m-1}(\Omega_s)^* \sin \theta_s e^{-i\phi_s} = 0 \]

\[
w_{nm}^\pm = \sqrt{(2n+1)(n \pm m)(n \mp m - 1) / (2n - 1)}
\]

\[
v_{nm}^\pm = \sqrt{(2n-1)(n \pm m)(n \pm m + 1) / (2n + 1)}
\]

Estimate the elevation using the arcsine function

\[ \theta_s = \sin^{-1}|\Phi_s| \]
Proposed method \( \text{sine-based recurrence relations} \)

- Sine-based recurrence relations of spherical harmonics

\[
2m Y_n^m(\Omega_s)^* + w_{nm}^+ Y_{n-1}^{m+1}(\Omega_s)^* \sin \theta_s e^{i\phi_s} + w_{nm}^- Y_{n-1}^{m-1}(\Omega_s)^* \sin \theta_s e^{-i\phi_s} = 0
\]

\[
2m Y_{n-1}^m(\Omega_s)^* + v_{nm}^+ Y_n^{m+1}(\Omega_s)^* \sin \theta_s e^{i\phi_s} + v_{nm}^- Y_n^{m-1}(\Omega_s)^* \sin \theta_s e^{-i\phi_s} = 0
\]

- Estimate the elevation using the arcsine function

\[
\theta_s = \sin^{-1}|\Phi_s|
\]

Overcome the singularity of EB–ESPRIT
Two types of recurrence relations of spherical harmonics

\[ 2m Y_n^m(\Omega_s)^* + w_{nm}^+ Y_{n-1}^{m+1}(\Omega_s)^* \sin \theta_s e^{i\phi_s} + w_{nm}^- Y_{n-1}^{m-1}(\Omega_s)^* \sin \theta_s e^{-i\phi_s} = 0 \rightarrow \text{Type 1} \]

- Type 1
  
  # of independent equations:
  
  \[ (N+1)^2 - 2 = 14 \]

  except \((n = m = 0)\) and \((n = 1, m = 0)\)
Two types of recurrence relations of spherical harmonics

\[
2mY^m_n(\Omega_s)^* + w^+_{nm} Y^{m+1}_{n-1}(\Omega_s)^* \sin \theta_s e^{i\phi_s} + w^-_{nm} Y^{m-1}_{n-1}(\Omega_s)^* \sin \theta_s e^{-i\phi_s} = 0 \quad \Rightarrow \text{Type 1}
\]

\[
2mY^m_{n-1}(\Omega_s)^* + v^+_{nm} Y^{m+1}_n(\Omega_s)^* \sin \theta_s e^{i\phi_s} + v^-_{nm} Y^{m-1}_n(\Omega_s)^* \sin \theta_s e^{-i\phi_s} = 0 \quad \Rightarrow \text{Type 2}
\]

- Type 1
  # of independent equations:
  \((N+1)^2 - 2 = 14\)

- Type 2
  # of independent equations:
  \(N^2 = 9\)

\(|m| \leq n \leq N\)
Two types of recurrence relations of spherical harmonics

\[ 2mY^m_n(\Omega_s)^* + w^{+}_{nm} Y^{m+1}_{n-1}(\Omega_s)^* \sin \theta_s e^{i\phi_s} + w^{-}_{nm} Y^{m-1}_{n-1}(\Omega_s)^* \sin \theta_s e^{-i\phi_s} = 0 \quad \rightarrow \text{Type 1} \]

\[ 2mY^m_{n-1}(\Omega_s)^* + v^+_{nm} Y^{m+1}_n(\Omega_s)^* \sin \theta_s e^{i\phi_s} + v^-_{nm} Y^{m-1}_n(\Omega_s)^* \sin \theta_s e^{-i\phi_s} = 0 \quad \rightarrow \text{Type 2} \]

- **Type 1**
  - \# of independent equations:
    - \((N+1)^2 - 2 = 14\)

- **Type 2**
  - \# of independent equations:
    - \(N^2 = 9\)

- **Redundant case**
  - \# of dependent equations:
    - \(N - 1 = 2\)

Total number of independent equations:

\[ 2N^2 + N = 14 + 9 - 2 = 21 \]
Proposed method | Number of detectable sources

- Two types of recurrence relations of spherical harmonics

\[ 2m Y^m_n (\Omega_s)^* + w_{nm}^+ Y^{m+1}_{n-1} (\Omega_s)^* \sin \theta e^{i\psi} + w_{nm}^- Y^{m-1}_{n-1} (\Omega_s)^* \sin \theta e^{-i\psi} = 0 \Rightarrow \text{Type 1} \]

\[ 2m Y^m_{n-1} (\Omega_s)^* + v_{nm}^+ Y^{m+1}_n (\Omega_s)^* \sin \theta e^{i\psi} + v_{nm}^- Y^{m-1}_n (\Omega_s)^* \sin \theta e^{-i\psi} = 0 \Rightarrow \text{Type 2} \]

- Type 1
  
  # of independent equations:
  \[(N + 1)^2 - 2 = 14\]

- Type 2
  
  # of independent equations:
  \[N^2 = 9\]

- Redundant case
  
  # of dependent equations
  \[N - 1 = 2\]

Total number of independent equations:

# of independent equations \(\Rightarrow\) # of detectable sources
Estimation procedure of sine-based EB-ESPRIT

Procedure of DOA estimation

Recurrence relation

\[ 2mY^m_n(\Omega_s) + w^+_{nm}Y^{m+1}_{n-1}(\Omega_s)^* \sin \theta e^{i\phi} + w^-_{nm}Y^{m-1}_{n-1}(\Omega_s)^* \sin \theta e^{-i\phi} = 0 \]

\[ 2mY^m_{n-1}(\Omega_s)^* + v^+_{nm}Y^{m+1}_n(\Omega_s)^* \sin \theta e^{i\phi} + v^-_{nm}Y^{m-1}_n(\Omega_s)^* \sin \theta e^{-i\phi} = 0 \]

Matrix form

Recurrence relation for \( Y^H \)

\[ 2MY^H(0,0) + W^+Y^H(-1,+1)\Phi_{\sin} + W^-Y^H(-1,-1)\Phi^*_\sin = 0 \]
\[ 2MY^H(-1,0) + V^+Y^H(0,+1)\Phi_{\sin} + V^-Y^H(0,-1)\Phi^*_\sin = 0 \]

Recurrence relation for \( \hat{U} \)

\[ 2M\hat{U}^{(0,0)} + W^+\hat{U}^{(-1,+1)}\Psi_{\sin} + W^-\hat{U}^{(-1,-1)}\Psi^*_\sin = 0 \]
\[ 2M\hat{U}^{(-1,0)} + V^+\hat{U}^{(0,+1)}\Psi_{\sin} + V^-\hat{U}^{(0,-1)}\Psi^*_\sin = 0 \]

\[ \hat{U} = Y^HT \]
Procedure of sine-based EB-ESPRIT

Stacking equations
\[
2[M \ M] \begin{bmatrix} \hat{U}^{(0,0)} \\ \hat{U}^{(-1,0)} \end{bmatrix} + [W^+ \hat{U}^{(-1,+1)} \ W^- \hat{U}^{(-1,-1)}] \begin{bmatrix} \Psi_{\text{sin}} \\ \Psi^*_{\text{sin}} \end{bmatrix} = 0
\]

\[
2MU + E\Psi_{\text{sin}} = 0
\]

Least squared solution
\[
\Psi_{\text{sin}} = -2E^\dagger MU
\]

Relationship between \(\Psi_{\text{sin}}\) and \(\Phi_{\text{sin}}\)
\[
\Psi_{\text{sin}} = T^{-1} \Phi_{\text{sin}} T
\]

Eigen value decomposition of matrix \(\Psi_{\text{sin}}\)

Matrix containing DOA information \(\Phi_{\text{sin}}\)
\[
\hat{\Theta}_{s}^{(d)} = \sin^{-1} \left| \Phi_{\text{sin}}^{(d)} \right|, \quad \hat{\phi}_{s}^{(d)} = \Delta \Phi_{\text{sin}}^{(d)}
\]
Procedure of sine-based EB-ESPRIT

Stacking equations

\[
2 \begin{bmatrix} \mathbf{M} & \mathbf{M} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{U}}^{(0,0)} \\ \hat{\mathbf{U}}^{(-1,0)} \end{bmatrix} + \begin{bmatrix} \mathbf{W}^+ & \hat{\mathbf{U}}^{(-1,+1)} \\ \mathbf{V}^+ & \hat{\mathbf{U}}^{(0,+1)} \end{bmatrix} \begin{bmatrix} \Psi_{\sin} \\ \Psi_{\sin}^* \end{bmatrix} = \mathbf{0} \]

\Rightarrow \quad 2\mathbf{M}\mathbf{U} + \mathbf{E}\Psi_{\sin} = \mathbf{0}

Solve the equation for \( \Psi_{\sin} \)

\[
\Psi_{\sin} = -2\mathbf{E}^\dagger\mathbf{M}\mathbf{U}
\]

Least squared solution

Relationship between \( \Psi_{\sin} \) and \( \Phi_{\sin} \)

\[
\Psi_{\sin} = \mathbf{T}^{-1}\Phi_{\sin}\mathbf{T}
\]

Eigen value decomposition of matrix \( \mathbf{M} \)

Matrix containing DOA information

\[
\hat{\Theta}_s^{(d)} = \sin^{-1}\begin{vmatrix} \Phi_{\sin}^{(d)} \end{vmatrix}, \quad \hat{\phi}_s^{(d)}
\]

\( \sin \theta \) vs angle(\( \theta \))
Procedure of sine-based EB-ESPRIT

Stacking equations

$2\begin{bmatrix} M & M \end{bmatrix} \begin{bmatrix} \hat{U}^{(0,0)} \\ \hat{U}^{(-1,0)} \end{bmatrix} + \begin{bmatrix} W^+ \hat{U}^{(-1,+1)} & W^- \hat{U}^{(-1,-1)} \\ V^+ \hat{U}^{(0,+1)} & V^- \hat{U}^{(0,-1)} \end{bmatrix} \begin{bmatrix} \Psi_{\sin} \\ \Psi_{\sin}^* \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$

$2MU + E\Psi_{\sin} = 0$

Solve the equation for $\Psi_{\sin}$

$\Psi_{\sin} = -2E^+MU$

Relationship between $\Psi_{\sin}$ and $\Phi_{\sin}$

$\Psi_{\sin} = T^{-1}\Phi_{\sin}T$

Eigen $\nu$

Matrix containing DOA information

$\hat{\Theta}_s^{(d)} = \sin^{-1}\left|\Phi_{\sin}^{(d)}\right|$, $\hat{\phi}_s^{(d)}$

Performance validation is needed
Performance validation

Simulation

| Objective | Background | Problem Statement | Proposed method | Performance validation |
Simulation for performance validation

Simulation configuration

- Number of microphones: 32 spherical t-design sampling, radius of sphere = 7.5 cm
- Maximum harmonics order: N = 3
- Target frequency: 1,456 Hz (kr = 2)
- Self-microphone noise (incoherent to signal)
- Compute covariance matrix: averaging 300 independent snapshots

Error measure of DOA estimation

- Estimation error
\[ \varepsilon^{(d)} = \left| \Omega^{(d)}_s - \hat{\Omega}^{(d)}_s \right|, \quad \Delta \theta^{(d)} = \left| \theta^{(d)}_s - \hat{\theta}^{(d)}_s \right|, \quad \Delta \phi^{(d)} = \left| \phi^{(d)}_s - \hat{\phi}^{(d)}_s \right| \]

- Signal-to-Noise Ratio (SNR)
\[ \text{SNR (dB)} = 10 \log_{10} \left( \frac{D \sigma^2_s}{L \sigma^2_n} \right), \]
\[ \sigma^2_s, \sigma^2_n: \text{power of signals and noises} \]

- Number of detectable sources
Validation for two incoherent sources

\[(\theta_s^{(1)}, \phi_s^{(1)}) = (20^\circ, 45^\circ), (\theta_s^{(2)}, \phi_s^{(2)}) = (46^\circ, 68^\circ)\]

\[\text{RMSE} = \sqrt{\frac{1}{JD} \sum_{j=1}^{J} \sum_{d=1}^{D} |E_j^{(d)}|^2}, \quad J = 400 \text{ independent trials}\]

Estimation performance is comparable to EB–ESPRIT
Simulation | RMSEs with various elevation angles

- **Single source:** $\phi_s = 60^\circ$, $0^\circ \leq \theta_s \leq 90^\circ$

- Validate the proposed method by comparing the RMSEs with various elevation angles and SNRs

$$\text{RMSE}(\theta) = \sqrt{\frac{1}{J_d} \sum_{j=1}^{J_d} \sum_{d=1}^{D} |\Delta \theta^{(d)}|^2}$$

$$\text{RMSE}(\phi) = \sqrt{\frac{1}{J_d} \sum_{j=1}^{J_d} \sum_{d=1}^{D} |\Delta \phi^{(d)}|^2}, \ J = 400 \ \text{independent trials}$$

![Graph showing errors in azimuth ($\Delta \phi_s$) vs. elevation ($\theta_s$)]
- Performance degradation in elevation
- Sine function slowly changes near $\theta_s = 90^\circ$
Simulation | RMSEs with various elevation angles

- Performance degradation in elevation
- Sine function slowly changes near $\theta_s = 90^\circ$

Despite performance degradation in elevation, a reasonable performance with error under $2^\circ$
Robustness measure: 2-norm condition number

\[ \Psi = -2E^\dagger M U \]

\[ CN_2(E) = \|E\|_2 \cdot \left\| (E^H E)^{-1} E^H \right\|_2 \]

Overcome the singularity problem

- Single source simulation
- Without self-microphone noise
- Condition number with different source elevation angles
RMSE variations with respect to the number of sources (D) and maximum order (N)

Number of detectable sources

\[ D_{\text{max}} \text{(Proposed)} = \left\lfloor N^2 + N / 2 \right\rfloor \geq D_{\text{max}} \text{(EB-ESRPIT)} = \left\lfloor N^2 / 2 \right\rfloor \]
Sine-based EB–ESPRIT utilizes two recurrence relations of spherical harmonics which has sine–based directional parameters

- Can estimate the DOAs near the equator without singularity
- No need for additional coordinate rotation
- Can estimate elevation and azimuth at once

More number of detectable sources than conventional EB–ESPRIT

- Increase of the number of independent equations
  → can estimate more number of sources simultaneously

Limitation

- Performance degradation in elevation near the equator due to the slow rate of change of sine function

Thank you
Byeongho Jo, KAIST
(byeongho@kaist.ac.kr)
