When can a System of Subnetworks be Registered Uniquely?

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Sensor network localization
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Need to deduce observers’ transforms
System of equations

GIVEN

Nodes: 1, \ldots, N \quad \text{(in } \mathbb{R}^d) \quad \text{ Nodes: 1, \ldots, N (in } \mathbb{R}^d) \quad \text{P}_i \subset \{1, \ldots, N\}

Patches: P_1, \ldots, P_M

x_{k,i}: \text{ local coordinate of node } k \text{ if } k \in P_i

\mathbf{x}_{k,i}: \text{ local coordinate of node } k \text{ if } k \in P_i

UNKNOWNS

\mathbf{z}_k: \text{ global coordinate of node } k
\mathcal{R}_i: \text{ rigid transform corresponding to } P_i, \text{ i.e. if } k \in P_i

\mathbf{z}_k = \mathcal{R}_i(\mathbf{x}_{k,i}) = \mathbf{O}_i \mathbf{x}_{k,i} + \mathbf{t}_i

Registration Problem

Find \mathbf{z}_1, \ldots, \mathbf{z}_N, \mathcal{R}_1, \ldots, \mathcal{R}_M \text{ such that}

\mathbf{z}_k = \mathcal{R}_i(\mathbf{x}_{k,i}), \quad k \in P_i, \quad i \in [1 : M]. \quad \text{(REG)}
System of equations

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z_k = \mathcal{R}_i(x_{k,i}), \quad k \in P_i, \quad i \in [1 : M]. \quad (\text{REG})
Registration Problem

Find $z_1, \ldots, z_N, R_1, \ldots, R_M$ such that

$$z_k = R_i(x_k, i), \quad k \in P_i, \quad i \in [1 : M].$$

(REG)
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Original network

<table>
<thead>
<tr>
<th>Observer</th>
<th>Node</th>
<th>Data</th>
<th>Global</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>$x_{1,1}$</td>
<td>$z_1$</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
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Given data
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Registration Problem

Find $z_1, \ldots, z_N, \mathcal{R}_1, \ldots, \mathcal{R}_M$ such that

$$z_k = \mathcal{R}_i(x_k, i), \quad k \in P_i, \quad i \in [1 : M]. \quad \text{(REG)}$$

- Does a solution exist?
  Yes! Ground truth

- Is this solution unique ... up to congruence?
  We are interested only in relative positions and transformations
Original network

Given data

Reconstructed network

(a)  

(b)  

(c)
Theorem: Uniqueness of solution

Suppose, for \( \text{REG} \) in \( \mathbb{R}^d \)

A1. each patch contains at least \( d + 1 \) nodes

A2. the nodes are in generic positions

Then

uniqueness of solution to \( \text{REG} \equiv \text{rigidity of the body graph} \)
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uniqueness of solution to $\text{REG} \equiv$ rigidity of the body graph

To test if this can be uniquely registered ...
Graph (embedding) rigidity: Setup

**GIVEN**

- undirected graph: $G = (V, E)$

- embedding of $G$ in $\mathbb{R}^d$: mapping $V \rightarrow \mathbb{R}^d$
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**QUESTION:** Can we have an embedding which preserves edge lengths, but has a different shape?
Graph (embedding) rigidity: Setup

Given an undirected graph $G = (V, E)$:

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**G** = \((\{1, 2, 3, 4\}, \{(1, 2), (1, 3), (1, 4), (2, 3), (3, 4)\})\)

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- Undirected graph: $G = (V, E)$
- Embedding of $G$ in $\mathbb{R}^d$: mapping $V \to \mathbb{R}^d$

**Question:** Can we have an embedding which preserves edge lengths, but has a different shape?

**Diagram:**

$G = \left( \{1, 2, 3, 4\}, \{(1, 2), (1, 3), (1, 4), (2, 3), (3, 4)\} \right)$

**QUESTION:** Can we have an embedding which preserves edge lengths, but has a different shape?
Graph (embedding) rigidity: Graph vs Embedding

\[ G = \left( \{1, 2, 3, 4, 5\}, \{(1, 2), (1, 4), (1, 5), (2, 3), (2, 5), (3, 4), (3, 5), (4, 5)\} \right) \]

This embedding is not rigid. But recall our theorem . . .

Suppose, for \( \text{reg} \) in \( \mathbb{R}^d \)

1. each patch contains at least \( d + 1 \) nodes
2. the nodes are in generic positions

Then uniqueness of solution to \( \text{reg} \) \( \equiv \) rigidity of the body graph.
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$$G = \left( \{1, 2, 3, 4, 5\}, \{(1, 2), (1, 4), (1, 5), (2, 3), (2, 5), (3, 4), (3, 5), (4, 5)\} \right)$$
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This embedding is rigid

Suppose, for \( \Delta \in \mathbb{R}^d \), each patch contains at least \( d + 1 \) nodes. Then uniqueness of solution to \( \Delta \equiv \) rigidity of the body graph.
Graph (embedding) rigidity: Graph vs Embedding

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Graph (embedding) rigidity: Generic embedding

Rigidity is a generic property

Given a graph, one of the following is true

- **every** generic embedding is rigid
- **every** generic embedding is non-rigid

Generic embedding $\implies$ rigidity becomes a **property of the graph**
To test if this can be uniquely registered ...
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Corollary for $d = 2$: REG in $\mathbb{R}^2$

Suppose, in a two-dimensional network

A1. each patch contains at least 3 nodes

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Then

uniqueness of solution to REG $\equiv$ 3-connectivity of the body graph
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Suppose, in a two-dimensional network

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uniqueness of solution to $\text{REG} \equiv$ 3-connectivity of the body graph

connected graph
\[ \exists \text{ path between every pair of vertices} \]

3-connected graph
remains connected if $\leq 3$ vertices removed
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existing tests for 2D rigidity: quadratic time
Summary

- Registration problem: assign global coordinates to points based on partial observations in different local coordinate systems related via rigid transforms.

- Focus: when is the solution unique.

- Under mild assumptions: uniqueness equivalent to rigidity of the body graph.

- Corollary for 2D networks: need only test 3-connectivity (linear time).
Thank You