Introduction

Motivation

- Tropical algebra [4] is an emerging and interesting field.
- Geometrical analysis of algorithms allows for intuition.
- Pruning naturally defines polytopes which enables geometrical analysis.

Contributions

- Analysing Viterbi and pruning in tropical algebra.
- Pruning occurs from the Cuninghame-Green inverse.
- Utilising objects of tropical geometry to better understand pruning.
- Metrics on polytopes.

Background

Tropical Algebra

- Similar to linear algebra, but the pair (+, x) is replaced by (\land, \cdot) (where \land = \min).
- Matrix/vector multiplication \[ [A \boxplus B]_i = \min_{j} \{A_i + B_j\} \]
- Neutral elements are 0 for the minimum and 0 for the addition.
- Example:
  \[ \begin{pmatrix} 2 & 4 \\ -6 & 11 \end{pmatrix} \boxplus \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} \min(2, 7, 4 + 3) \\ \min(-6, 7, 11 + 3) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \]

Tropical Geometry

Definition 1: Let \( a, b \in \mathbb{R}^{n+1}_\min \). An affine tropical half-space is a set of \( \mathbb{R}^{n+1}_\min \) defined by:
\[ \Theta(a, b) := \{ x \in \mathbb{R}^{n+1}_\min \mid a \cdot x + b, a \cdot \min(x) \leq b \} \]

- Tropical polyhedra are intersections of affine tropical half-spaces.
- Tropical polytopes are bounded tropical polyhedra.

Tropical Viterbi

- Viterbi algorithm [3]:
\[ q(t) = \left( \max_{i} w_i g(t - 1) \right) \cdot b_i(s) \tag{1} \]
- Negative logarithm of (1) and \( x(t) = -\log q(t) \), \( A = -\log W \), \( p(s) = -\log b(s) \):
\[ \bar{x}(t) = A^{\top} \boxplus \bar{x}(t - 1) + p(s) \tag{2} \]
- Define \( P(s) \):
\[ P(s) = \begin{bmatrix} p_0(s) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & p_n(s) \end{bmatrix} \]
- Viterbi in tropical algebra:
\[ x(s) = P(s) \boxplus A^{\top} \boxplus \bar{x}(t - 1) \tag{3} \]
- Pruning: go through \( x(t) \) and set values greater than a threshold to +\( \infty \).
- Indices that should be pruned
  \[ \text{Cuninghame-Green inverse} \]

Proposition 1: Let
\[ X(t) = \begin{bmatrix} x_0(t) & \cdots & x_n(t) \end{bmatrix} \]
where \( x_i(t) \) represents the \( i \)-th element of the vector \( x(t) \), and let \( \eta = \theta + \frac{1}{2} \{ \bar{x}(t) \boxplus \bar{x}(t) \} + \Theta \), where \( \Theta \) is a vector that comprises of \( 0 \) and \( \bar{\Theta} \) is the kerenity variable. Finally, let \( \boxplus \) denote the max-plus matrix multiplication and \( X(t) = X(t) \boxplus \bar{x}(t) \). Then, the negative elements of \( X(t) \) indicate which indices of \( x(t) \) need to be pruned.

\[ x_i(t) = \begin{bmatrix} x_0(t) & \cdots & x_n(t) \end{bmatrix} \]
\[ \text{Polytope} \]
\[ n \times 1 \text{ - faces} \]

Geometry of the Viterbi

- Variable vector \( x \)
  - from below:
  \[ x \geq b, \quad b = P(s) \boxplus A^{\top} \boxplus \bar{x}(t - 1) \]
- from above:
\[ x \leq \eta, \quad \eta = \theta + \frac{1}{2} \{ b^{\top} \boxplus b \} + \Theta \]

- \((5) + (6)\)
- \((5) - (6)\) - faces
- \( \text{best paths} \)

Example and Experimentation

Numerical Example for Weighted Finite State Transducers

Transition matrix \( A \), observation matrix \( P(z) \):
\[ A = \begin{bmatrix} 0.523 & 0.523 & 0.523 \\ 0.523 & 0.523 & 0.523 \\ 0.523 & 0.523 & 0.523 \end{bmatrix}, \quad P(z) = \begin{bmatrix} 0.523 & 0.523 & 0.523 \\ 0.523 & 0.523 & 0.523 \\ 0.523 & 0.523 & 0.523 \end{bmatrix} \]

- Starting state \( x(0) \):
\[ x(0) = [0 \ 0 \ 0 \ 0 \ 0] \]

- Outputs:
\[ x(1) = [1.125 \ 1.28 \ 1.581 \ 1.28 \ 0 \ 0 \] \]
\[ x(2) = [1.28 \ 0 \ 0 \ 0 \ 0 \ 0] \]
\[ x(3) = [1.28 \ 0 \ 0 \ 0 \ 0 \ 0] \]

- Polytopes for \( x(1) \) and \( x(2) \):

\[ \text{NLP Experiment & Application} \]

- Transliteration task: greeklish (latin text) vs greek text
- Minimising derivative of \( \varepsilon \) vs maximising \( \lambda \)
- Best results:

\[ \theta = 10 \]

Geometry of the Viterbi

- Variable vector \( x \)
  - from below:
  \[ x \geq b, \quad b = P(s) \boxplus A^{\top} \boxplus \bar{x}(t - 1) \]
  - from above:
\[ x \leq \eta, \quad \eta = \theta + \frac{1}{2} \{ b^{\top} \boxplus b \} + \Theta \]

- \((5) + (6)\)
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- \( \text{best paths} \)

References