Minimax Game-Theoretic Approach to Multiscale H-infinity Optimal Filtering

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Outline

- Multiscale systems
- State-space dyadic tree model
- Estimator Design
- Multiscale H-infinity Filter
- Implementation Results
  - H-infinity vs Kalman Filter
  - Estimation Error
- Conclusion
Multiscale systems

Applications:
- Data fusion in Demographics
- Terrain Mapping using UAV
- Video streaming/encoding
- Efficiency in recursive parallelizable algorithms

For 2D images applications: just use **quad** instead of **dyadic trees**
Multiresolution Signal Example
State-Space Models on Dyadic Trees

Given a multiresolution representation of a signal, we relate levels of the tree through a state-space model.

Forward System Dynamics:

\[
\begin{align*}
x_{k+1}^{m\alpha} &= A_{k+1}^{m\alpha} x_k^m + B_{k+1}^{m\alpha} w_{k+1}^m \\
x_{k+1}^{m\beta} &= A_{k+1}^{m\beta} x_k^m + B_{k+1}^{m\beta} w_{k+1}^m \\
y_k^m &= C_k^m x_k^m + v_k^m
\end{align*}
\]

\(k = 0, 1, \ldots, K; \ m = 1, 2, \ldots, M_k\)

\(x_0^1 = x_0\)

\(w_k^m \sim N(0, I), \ v_k^m \sim N(0, R_k^m)\)

- Interpolation and higher resolution detail

Note: noise signals need to be \(l_2\) bounded.

Depiction of the dyadic tree structure for a multiscale linear system.
Example: Multiresolution Signal

Stage 0: \( x_0^1 \sim N(0, p_0) \in \mathbb{R} \)  
\( A = B = C = 1 \)  
\( w_k \sim N(0, 1/2^{K-k}) \)  
\( v_k \sim N(0, 1/(K-k)) \)

\[
x_{k+1}^{\alpha} = A_{k+1}^{\alpha} x_k^m + B_{k+1}^{\alpha} w_{k+1}^m \\
x_{k+1}^{\beta} = A_{k+1}^{\beta} x_k^m + B_{k+1}^{\beta} w_{k+1}^m \\
y_k^m = C_k^m x_k^m + v_k^m
\]
Key features

- Coarse-to-fine recursion: Multiresolution synthesis of signals

- A resolution level captures the features of signals up to that level that are relevant for finer prediction downwards

- Similar to a dynamical system but state evolution is spatial instead of temporal
State-Estimation

$H_\infty$ Estimator Design Problem for Multiscale Systems
Problem: Estimator Design

“Given exogenous noisy measurements at various resolution levels, design an optimal filter for state-estimation.”

System:

\[
\begin{align*}
x_{k+1}^{m\alpha} &= A_{k+1}^{m\alpha} x_k^m + B_{k+1}^{m\alpha} w_{k+1}^{m\alpha} \\
x_{k+1}^{m\beta} &= A_{k+1}^{m\beta} x_k^m + B_{k+1}^{m\beta} w_{k+1}^{m\beta} \\
y_k^m &= C_k^m x_k^m + v_k^m
\end{align*}
\]  

(1)

Estimate:

\[
\hat{z}_k^m = L_k^m x_k^m
\]  

(2)

Given attenuation level, minimize the supremum of quadratic cost subject to (1)

\[
J = \frac{\sum_{k=1}^{K-1} \sum_{m=1}^{M_k} ||z_k^m - \hat{z}_k^m||_Q^m}{||x_0 - \hat{x}_0||_{P_0}^2 + \sum_{k=1}^{K-1} \sum_{m=1}^{M_k} \left\{ ||w_{k+1}^{m\alpha}||_I^2 + ||w_{k+1}^{m\beta}||_I^2 + ||v_k^m||_{R_{k-1}^{m-1}}^2 \right\}}
\]

\[
sup J < 1/\gamma.
\]
Solution Methodology

• Equivalent Minimax problem

\[
\min_{\hat{x}_k^m} \max_{(y_k^m, w_k^m, x_0)} J = \frac{1}{2} \sum_{k=1}^{K-1} \sum_{m=1}^{M_k} \{||x_k^m - \hat{x}_k^m||^{2}_{Q_k^m} - \frac{1}{\gamma} (||w_{k+1}^m||^{2}_{I})
\]

\[
+ ||w_{k+1}^m||^{2}_{I} + ||y_k^m - C_k^m x_k^m||^{2}_{R_{k-1}^m}) \} - \frac{1}{2\gamma} ||x_0 - \hat{x}_0||^{2}_{p_{0-1}}
\]

subject to (1) and (2) where, \( \tilde{Q}_k^m = L_k^m L_k^m \)

• First, solve for optimal \( x_0 \) and \( w_k \) by constructing Hamiltonian (Lagrange multipliers) and using Maximum Principle

• Solution assumption:

\[
x_k^m* = \tilde{x}_k^m + P_k^m \lambda_k^m* \text{. where } \lambda_k^m* \text{ are the Lagrange multipliers}
\]
Estimator Design

- This gives optimal values for $x_0$ and $w_k$

$$\begin{bmatrix} w_{k+1}^{m\alpha*} \\ w_{k+1}^{m\beta*} \end{bmatrix} = \begin{bmatrix} B_{k+1}^{m\alpha T} \\ B_{k+1}^{m\beta T} \end{bmatrix} \begin{bmatrix} \lambda_{k+1}^{m\alpha*} \\ \lambda_{k+1}^{m\beta*} \end{bmatrix}, \quad x_0^* = \hat{x}_0 + p_0\lambda_0^*.$$

- Simplifying the cost with above optimal values reduces the problem to

$$\min \max J = \frac{1}{2} \sum_{k=1}^{K-1} \sum_{m=1}^{M_k} \left[ \frac{1}{Q_k^m} - \frac{1}{\gamma} (\|y_k^m - C_k^m \bar{x}_k^m\|^2 - \|\bar{x}_k^m - \hat{x}_k^m\|^2 \right]$$

subject to two coupled equations for two children nodes

$$\begin{align*}
\bar{x}_{k+1}^m - A_{k+1}^{m\alpha} \bar{x}_k^m - A_{k+1}^{m\alpha} P_k^m (I - \gamma Q_k^m P_k^m + C_k^m R_k^m C_k^m P_k^m)^{-1} [\gamma Q_k^m (\bar{x}_k^m - \hat{x}_k^m) + C_k^m R_k^m (y_k^m - C_k^m \bar{x}_k^m)] = \\
- P_{k+1}^{m\alpha} \lambda_{k+1}^{m\alpha*} + A_{k+1}^{m\alpha} P_k^m (I - \gamma Q_k^m P_k^m + C_k^m R_k^m C_k^m P_k^m)^{-1} [\gamma Q_k^m (\bar{x}_k^m - \hat{x}_k^m) + C_k^m R_k^m (y_k^m - C_k^m \bar{x}_k^m)] = 0,
\end{align*}$$

$$\begin{align*}
\bar{x}_{k+1}^m - A_{k+1}^{m\beta} \bar{x}_k^m - A_{k+1}^{m\beta} P_k^m (I - \gamma Q_k^m P_k^m + C_k^m R_k^m C_k^m P_k^m)^{-1} [\gamma Q_k^m (\bar{x}_k^m - \hat{x}_k^m) + C_k^m R_k^m (y_k^m - C_k^m \bar{x}_k^m)] = \\
- P_{k+1}^{m\beta} \lambda_{k+1}^{m\beta*} + A_{k+1}^{m\beta} P_k^m (I - \gamma Q_k^m P_k^m + C_k^m R_k^m C_k^m P_k^m)^{-1} [\gamma Q_k^m (\bar{x}_k^m - \hat{x}_k^m) + C_k^m R_k^m (y_k^m - C_k^m \bar{x}_k^m)] = 0.
\end{align*}$$
Estimator Design (contd.)

• Coupled equality constraints are expressed as a matrix equation

\[
\min_{\hat{x}_k^m} \max_{y_k^m} J = \frac{1}{2} \sum_{k=1}^{K-1} \sum_{m=1}^{M_k} \left[ ||\bar{x}_k^m - \hat{x}_k^m||_Q^m - \frac{1}{\gamma} (||y_k^m - C_k^m \bar{x}_k^m||_{R_k^m}^2) \right]
\]

subject to

\[
\begin{bmatrix}
P_{k+1}^{m\alpha} & P_{k+1}^{m\beta} \\
\end{bmatrix}
\begin{bmatrix}
\lambda_{k+1}^{m\alpha*} \\
\lambda_{k+1}^{m\beta*} \\
\end{bmatrix}
= 
\begin{bmatrix}
*_1 & *_2 \\
*_2^T & *_3 \\
\end{bmatrix}
\begin{bmatrix}
\lambda_{k+1}^{m\alpha*} \\
\lambda_{k+1}^{m\beta*} \\
\end{bmatrix}
\]

• Must hold for arbitrary lambda, so we arrive at matrix equality constraint:

\[
\begin{bmatrix}
P_{k+1}^{m\alpha} - *_1 \\
P_{k+1}^{m\beta} - *_3 \\
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
\end{bmatrix}
\]

• This gives us a two-person zero-sum game having unique saddle-point equilibrium strategy

\[
\implies \hat{x}_k^{m*} = \bar{x}_k^m, \ y_k^{m*} = C_k^m \bar{x}_k^m
\]
Estimator Design (contd.)

• The optimal strategy for measurement noise follows directly

\[ v_k^{m*} = y_k^{m*} - C_k^{m*} \hat{x}_k^{m*} = C_k^{m*} \bar{x}_k^{m*} - C_k^{m*} \bar{x}_k^{m*} = 0. \]

• Hence, the optimal H-infinity filter is given as:

\[ \hat{z}_k^{m*} = L_k^{m*} \hat{x}_k^{m*}; \quad k = 0 \ldots K, \ m = 1 \ldots M_k \]

where

\[ \hat{x}_{k+1}^{m\alpha*} = A_{k+1}^{m\alpha} \hat{x}_k^{m*} + K_{k+1}^{m\alpha} (y_k^m - C_k^{m*} \hat{x}_k^{m*}), \]

\[ \hat{x}_{k+1}^{m\beta*} = A_{k+1}^{m\beta} \hat{x}_k^{m*} + K_{k+1}^{m\beta} (y_k^m - C_k^{m*} \hat{x}_k^{m*}), \]

\[ K_{k+1}^{m\alpha} = A_{k+1}^{m\alpha} P_k^{m*} (I - \gamma \bar{Q}_k^{m} P_k^{m} + C_k^{m T} R_k^{-1} C_k^{m} P_k^{m})^{-1} C_k^{m T} R_k^{-1}, \]

\[ K_{k+1}^{m\beta} = A_{k+1}^{m\beta} P_k^{m*} (I - \gamma \bar{Q}_k^{m} P_k^{m} + C_k^{m T} R_k^{-1} C_k^{m} P_k^{m})^{-1} C_k^{m T} R_k^{-1}. \]

• And \( P_k^m \) is determined by a Joint Difference Riccati Equation
Theorem:

For noise attenuation $\gamma > 0$, an $H_\infty$ filter for $x^m_k$ exists if and only if there exists a stabilizing solution $P^m_k > 0 \ \forall k, m$ to the coupled-pair of discrete-time Riccati equations:

$$P^{m,\alpha}_{k+1} = A^{m,\alpha}_{k+1} P^m_k (I - \gamma \tilde{Q}^m_k P^m_k + C^m_k R^{m-1}_k C^m_k P^m_k)^{-1} A^{m,\alpha T}_{k+1} + B^{m,\alpha}_{k+1} B^{m,\alpha T}_{k+1},$$

$$P^{m,\beta}_{k+1} = A^{m,\beta}_{k+1} P^m_k (I - \gamma \tilde{Q}^m_k P^m_k + C^m_k R^{m-1}_k C^m_k P^m_k)^{-1} A^{m,\beta T}_{k+1} + B^{m,\beta}_{k+1} B^{m,\beta T}_{k+1},$$

$$P_0 = P_0.$$

The $H_\infty$ filter is given by,

$$\hat{x}^{m,\alpha*}_{k+1} = A^{m,\alpha}_{k+1} \hat{x}^m_k + K^{m,\alpha}_{k+1} (y^m_k - C^m_k \hat{x}^m_k),$$

$$\hat{x}^{m,\beta*}_{k+1} = A^{m,\beta}_{k+1} \hat{x}^m_k + K^{m,\beta}_{k+1} (y^m_k - C^m_k \hat{x}^m_k),$$

$$K^{m,\alpha}_{k+1} = A^{m,\alpha}_{k+1} P^m_k (I - \gamma \tilde{Q}^m_k P^m_k + C^m_k R^{m-1}_k C^m_k P^m_k)^{-1} C^m_k R^{m-1}_k,$$

$$K^{m,\beta}_{k+1} = A^{m,\beta}_{k+1} P^m_k (I - \gamma \tilde{Q}^m_k P^m_k + C^m_k R^{m-1}_k C^m_k P^m_k)^{-1} C^m_k R^{m-1}_k.$$

- A special case is when $\gamma \to 0$ the filter reduces to standard Kalman filter.

Proof was just described earlier.

Current H-infinity Estimator [2]

- To take into account measurements at current stage
- For a simple linear system, the current H-infinity estimator is given as:

System:  
\[
x_{k+1} = A_k x_k + B_k w_k \\
y_k = C_k x_k + v_k \\
z_k = L_k x_k
\]

State estimate error:  
\[
e_k = \hat{z}_{k|k} - L_k x_k
\]

Note that \(z_{k|k-1}\) is replaced by \(z_{k|k}\) here. The rest of cost function is the same.

Filter:  
\[
\hat{z}_k = L_k \hat{x}_k \\
\hat{x}_{k|k-1} = A_k \hat{x}_{k-1|k-1} \\
P_{k|k-1} = A_{k-1} P_{k-1|k-1} A_{k-1}^T + B_{k-1} B_{k-1}^T \\
\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k (y_k - C_k x_{k|k-1}) \\
K_k = P_{k|k-1} C_k^T (R + C_k P_{k|k-1} C_k^T)^{-1} \\
P_{k|k} = P_{k|k-1} - P_{k|k-1} \left[ C_k^T \; L_k^T \right] R_{e,k}^{-1} \left[ C_k \; L_k \right] P_{k|k-1} \\
R_{e,k} = \begin{bmatrix} R & 0 \\ 0 & -I/\gamma \end{bmatrix} + \begin{bmatrix} C_k \\ L_k \end{bmatrix} P_{k|k-1} \begin{bmatrix} C_k^T \\ L_k^T \end{bmatrix}
\]

Combining the above result with predictor-based H-infinity design proposed earlier, gives the current-H-infinity-estimator – completely written in paper.

Note: For simplicity, we’ve avoided writing it down here.

Results and Comparison

- Kalman vs. H-infinity Filter
- Estimation error trend along stages
Gray 70% confidence bounds indicate the uncertainty in state transitions. Observations are indicated by black dots, while the original signal is in blue. State estimates recovered by proposed filter are indicated by red dotted line.
Working with Images: The Signal
Working with Images: The Model
Working with Images: The Estimation

6 pairs showing Original (left) and Estimated (right) images of each of the 6 stages
Working with Images: Current Estimator

Original

Current-Estimator $\gamma = 185.44$ | Signal-to-noise ratio: 26.3 dB (HF), 19.3 dB (KF)
Comparison of SNR
*Stage 4 was unobserved, hence the estimates (images on right) of stage 4 and 5 are identical*
Covariance trend across stages
Conclusion

• Presented a minimax game-theoretic solution to multiscale optimal estimation problem.  
  *This solution avoids the need to separately solve smoothing and filtering problems which has been the classical approach in multiscale recursive estimation [3, 4].*

• Reduction of 21% in the estimation cost observed by using H-infinity filter instead of Kalman filter.  
  *Estimation cost is calculated by accumulating squared differences between original and state estimates.*

• High SNR value for H-infinity filter estimates in comparison to Kalman filter shown for all stages of multiresolution signal.

• State-estimate-recovery properties highlighted in the experiments exhibiting missing observations.

• Robust to worst-case additive exogenous noise.

• Experimental results corroborate theoretical findings; evaluations shown for 1-D and 2-D signal examples.