CONTROL OF GRAPH SIGNALS OVER RANDOM TIME-VARYING GRAPHS

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Control of Graph Signals

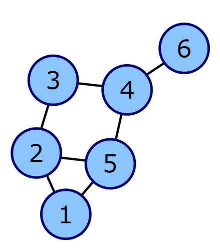
- Graph Signal Processing (GSP)
  - ⇒ Process signals defined on nodes of a graph (graph signals)
  - ⇒ Exploit information contained in the underlying graph structure
- Control graph signal diffusion ⇒ Drive signal to desired state
- Act only on a few relevant nodes ⇒ Control nodes
  - ⇒ Sparse controllability of graph signals [Segarra '16, Barbarossa '16]
- Random graphs ⇒ Probability of link failure
  - ⇒ Link or sensor failure in the grid, street closures

Objective

- Drive diffusion of graph signals to a desired state
  - ⇒ Act on a few preselected control nodes
  - ⇒ Design appropriate control signals
- Drive to a bandlimited state by means of bandlimited control signals
- Incorporate stochastic nature of underlying support
  - ⇒ Introduce concept of controllability in the mean
  - ⇒ Mean square error analysis

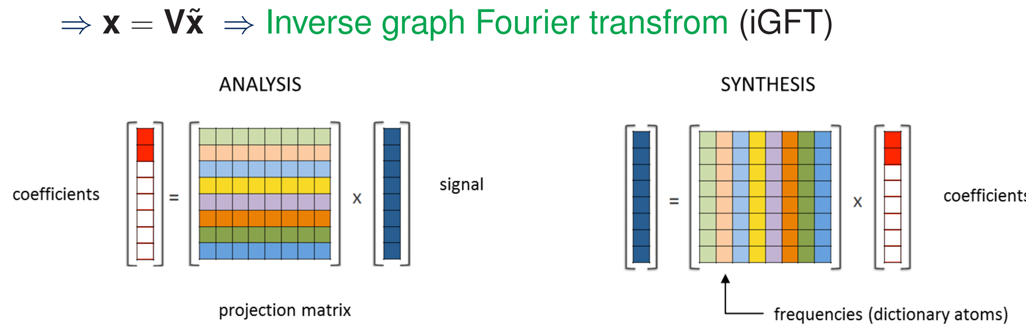
Graph signals

- Weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$ with n nodes
- Graph signal $\mathbf{x} \in \mathbb{R}^n$ ⇒ Data value on each node
- Graph shift operator $\mathbf{S} \in \mathbb{R}^{n \times n}$ ⇒ Captures local structure in $\mathcal{G}$



$$\mathbf{S} = \begin{pmatrix} S_{11} & S_{12} & 0 & 0 & S_{15} & 0 \\ S_{21} & S_{22} & S_{23} & 0 & S_{25} & 0 \\ 0 & S_{23} & S_{33} & S_{34} & 0 & 0 \\ 0 & 0 & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & 0 & S_{54} & S_{55} & 0 \\ 0 & 0 & S_{64} & 0 & 0 & S_{66} \end{pmatrix}$$

- Interaction between signal and support ⇒ $\mathbf{Sx}$ local operation
- Focus on graph Laplacian $\mathbf{S} = \mathbf{L} = \mathbf{D} - \mathbf{W}$ ($\mathbf{D}$: degree, $\mathbf{W}$: adjacency)
- Graph Laplacian is symmetric and positive semidefinite
  - ⇒ Orthogonal eigendecomposition of GSO $\mathbf{S} = \mathbf{L} = \mathbf{V \Lambda V}^H$
- Project graph signal onto eigenbasis ⇒ $\tilde{\mathbf{x}} = \mathbf{V}^H\mathbf{x}$
  - ⇒ Defined as the graph Fourier transform (GFT)
- Linear combination of eigenvectors weighted by GFT coefficients
  - ⇒ $\mathbf{x} = \mathbf{V}\tilde{\mathbf{x}}$ ⇒ Inverse graph Fourier transfrom (iGFT)



- Bandlimited graph signal ⇒ $\tilde{\mathbf{x}} = [\tilde{\mathbf{x}}_K^T, \mathbf{0}_{N-K}^T]^T$, $\mathbf{V} = [\mathbf{V}_K, \mathbf{V}_{N-K}]$
  - ⇒ Sparse representation in the graph frequency domain

References

- S. Segarra et al., "Reconstruction of graph signals through percolation of seeding nodes," *IEEE TSP*, Aug. 2016.
- S. Barbarossa et al., "On sparse controllability of graph signals," *IEEE ICASSP*, March 2016.
- R. Varma et al., "Spectrum-blind signal recovery on graphs," *IEEE CAMSAP*, Dec. 2015.
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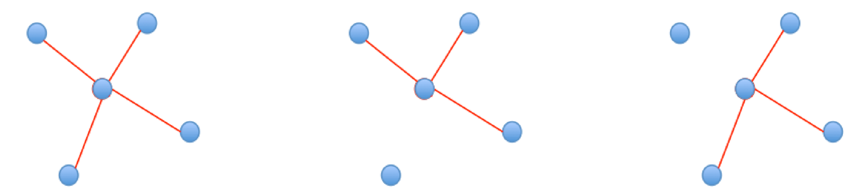
Random Graph Model



- Random Edge Sampling (RES)
  - ⇒ Edge (i,j) is active with probability p_{ij}
- Underlying graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ ⇒ Graph realization $\mathcal{G}_t = (\mathcal{V}, \mathcal{E}_t)$
  - ⇒ Edge set $\mathcal{E}_t \subseteq \mathcal{E}$ ⇒ $\mathbb{P}((i,j) \in \mathcal{E}_t) = p_{ij}$ (if $(i,j) \in \mathcal{E}$, 0 otherwise)
- Edges are activated independently
  - ⇒ Activation probability $p_{ij} = p_{\text{act}}$ for all $(i,j) \in \mathcal{E}$
- $\mathbf{L}$: graph Laplacian of underlying graph $\mathcal{G}$
- $\mathbf{L}_t$: graph Laplacian of realization $\mathcal{G}_t$
- $\bar{\mathbf{L}} = \mathbb{E}[\mathbf{L}_t] = p_{\text{act}}\mathbf{L}$: graph Laplacian of expected graph $\bar{\mathcal{G}}$
  - ⇒ For $\|\mathbf{L}\| \leq \varrho$ ⇒ $\|\mathbf{L}_t\| \leq \|\mathbf{L}\| \leq \varrho$ (interlacing property)

Diffusion Control over Graphs

- Diffusion signal $\mathbf{x}_t$ for $t = 1, 2, \ldots$

$$\mathbf{x}_{t+1} = (\mathbf{I} - \epsilon\mathbf{L})\mathbf{x}_t := \mathbf{A}\mathbf{x}_t$$

  - ⇒ $\mathbf{x}_0$: initial state; $\mathbf{A}$: state transition matrix, $0 < \epsilon \leq 1/\varrho$ (stability)
- Steer from $\mathbf{x}_0$ to a desired state $\mathbf{x}^*$ in $T < \infty$
  - ⇒ Design control signals $\mathbf{u}_t$ to act on the nodes $\mathcal{V}$

$$\mathbf{x}_{t+1} = \mathbf{A}\mathbf{x}_t + \mathbf{B}\mathbf{u}_t$$

  - ⇒ $\mathbf{B}$: control input matrix
- Objective: design $\mathbf{u}_t$ such that $\mathbf{x}_T = \mathbf{x}^*$
- Design sparse control signals that minimize energy [Barbarossa '16]
  - ⇒ No fixed set of control nodes over time
- Preselect nodes, design control inputs, apply graph filter [Segarra '16]
  - ⇒ No stochasticity in the underlying graph support

Sparse Control in the Graph Fourier Domain

- Drive system to a desired bandlimited state $\mathbf{x}^* = \mathbf{V}_K^H\tilde{\mathbf{x}}_K^*$
- Control signals on a subset of $M \leq N$ nodes $\mathcal{S} \subseteq \mathcal{V}$
  - ⇒ Bandlimited control signals ⇒ $\mathbf{u}_t = \mathbf{V}_K^H\tilde{\mathbf{u}}_{t,K}$
- Study control system in the graph frequency domain

$$\tilde{\mathbf{x}}_{t+1} = \tilde{\mathbf{A}}\tilde{\mathbf{x}}_t + \mathbf{V}^H\text{diag}(\mathbf{d})\mathbf{u}_t$$

  - ⇒ $\tilde{\mathbf{A}} = \mathbf{V}^H\mathbf{A}\mathbf{V}$ ⇒ $\mathbf{A}$ shares eigenvectors with $\mathbf{L}$ ⇒ $\tilde{\mathbf{A}} = \text{diag}(\tilde{\mathbf{a}})$
  - ⇒ $\mathbf{d}$: selection vector ⇒ $[\mathbf{d}]_i = 1$ if node $i \in \mathcal{S}$
- Rewrite system separating K desired frequencies

$$\begin{bmatrix} \tilde{\mathbf{x}}_{t+1,K} \\ \tilde{\mathbf{x}}_{t+1,N-K} \end{bmatrix} = \begin{bmatrix} \text{diag}(\tilde{\mathbf{a}}_K)\tilde{\mathbf{x}}_{t,K} \\ \text{diag}(\tilde{\mathbf{a}}_{N-K})\tilde{\mathbf{x}}_{t,N-K} \end{bmatrix} + \mathbf{V}^H\text{diag}(\mathbf{d})\mathbf{V}_K\tilde{\mathbf{u}}_{t,K}$$

  - ⇒ $\mathbf{u}_t$ is bandlimited so $\tilde{\mathbf{u}}_{t,N-K} = \mathbf{0}$
- Design selection $\mathbf{d}$ so that $(\text{diag}(\tilde{\mathbf{a}}), \mathbf{V}^H\text{diag}(\mathbf{d})\mathbf{V}_K)$ is controllable
  - ⇒ Any output can be obtained from bandlimited control signals
- Selection $\text{diag}(\mathbf{d})$ ⇒ Evolution affects all frequencies coefficients
  - ⇒ System states cannot be bandlimited graph signals for all t
- Drive only the K desired frequencies ⇒ Design $\text{diag}(\mathbf{d})$ and $\tilde{\mathbf{u}}_{t,K}$

$$\tilde{\mathbf{x}}_{t+1,K} = \text{diag}(\tilde{\mathbf{a}}_K)\tilde{\mathbf{x}}_{t,K} + \mathbf{V}_K^H\text{diag}(\mathbf{d})\mathbf{V}_K\tilde{\mathbf{u}}_{t,K}$$

  - ⇒ Filter out the non-desired frequency content
  - ⇒ $\mathbf{x}^* = \mathbf{H}\mathbf{x}_T$ such that $\mathbf{V}^H\mathbf{x}^* = \mathbf{V}^H\mathbf{H}\mathbf{x}_T = [(\tilde{\mathbf{x}}_K^*)^T, \mathbf{0}_{N-K}^T]^T$

Proposition

A **necessary condition** on the number of control nodes to drive $\mathbf{x}_t$ to a desired bandlimited state $\tilde{\mathbf{x}}^* = \mathbf{V}_K\tilde{\mathbf{x}}_K^*$ in the graph Fourier domain is that at least

$$M \geq \frac{K}{T}$$

nodes must be selected to inject the input signal into the system.

- The longer the time considered, the less nodes needed to control

Controllability in the Mean

- Given the random edge sampling (RES) graph model
- Time-varying control system ⇒ $\mathbf{A}_t$ depends on the changing topology

$$\mathbf{x}_{t+1} = \mathbf{A}_t\mathbf{x}_t + \text{diag}(\mathbf{d})\mathbf{u}_t$$

  - ⇒ Selected nodes in $\mathbf{d}$ are constant for all t
- Control the mean evolution of the system
  - ⇒ $\mathbf{x}_t$ depends on $\mathbf{A}_\tau$ for $\tau = 0, \ldots, t-1$ ⇒ Independent of $\mathbf{A}_t$

$$\boldsymbol{\mu}_{t+1} = \mathbb{E}[\mathbf{x}_{t+1}] = \mathbb{E}[\mathbf{A}_t]\mathbb{E}[\mathbf{x}_t] + \text{diag}(\mathbf{d})\mathbf{u}_t = \bar{\mathbf{A}}\boldsymbol{\mu}_t + \text{diag}(\mathbf{d})\mathbf{u}_t$$

  - ⇒ Constant activation ⇒ $\bar{\mathbf{A}} = \mathbf{I} - \epsilon p_{\text{act}}\mathbf{L}$ ⇒ $\bar{\mathbf{A}} = \mathbf{V}\text{diag}(\bar{\mathbf{a}})\mathbf{V}^H$
- Mean evolution ⇒ Deterministic diffusion control system
  - ⇒ Drive the system in the frequency domain ⇒ $\tilde{\boldsymbol{\mu}}_t = \mathbf{V}^H\boldsymbol{\mu}_t$
  - ⇒ Focus on K frequencies ⇒ $\tilde{\boldsymbol{\mu}}_t = [\tilde{\boldsymbol{\mu}}_{t,K}^T, \tilde{\boldsymbol{\mu}}_{t,N-K}^T]^T$

$$\tilde{\boldsymbol{\mu}}_{t+1,K} = \text{diag}(\bar{\mathbf{a}}_K)\tilde{\boldsymbol{\mu}}_{t,K} + \mathbf{V}_K^H\text{diag}(\mathbf{d})\mathbf{V}_K\tilde{\mathbf{u}}_{t,K}$$

- Drive the mean signal to a desired bandlimited signal $\tilde{\mathbf{x}}_K^*$ ⇒ $\tilde{\boldsymbol{\mu}}_{T,K} = \tilde{\mathbf{x}}_K^*$
  - ⇒ Filter out the non-desired frequency content ⇒ $\mathbf{H}\boldsymbol{\mu}_T$
  - ⇒ Desired K frequency content *in the mean*

Mean Square Analysis

- Mean square analysis to study robustness of the adopted control

Proposition

Assume $\mathbf{x}_0 = \mathbf{0}$, then

$$\mathbb{E}\left[\|\mathbf{x}_T - \boldsymbol{\mu}_T\|^2\right] \leq \sum_{\tau=0}^{T-1}\sum_{\tau'=0}^{T-1} \text{tr}\left[\text{diag}(\mathbf{d})\mathbf{u}_\tau\mathbf{u}_{\tau'}\right]$$

- Bound depends on the design variables through $\mathbf{d}$ and $\mathbf{u}_\tau$

Corollary

Define $\mathbf{U}_K = [\tilde{\mathbf{u}}_{0,K}, \ldots, \tilde{\mathbf{u}}_{T-1,K}] \in \mathbb{C}^{K \times T}$ and $\mathbf{1}_T$ is the all-one vector of size T. Then,

$$\mathbb{E}\left[\|\mathbf{x}_T - \boldsymbol{\mu}_T\|^2\right] \leq \|\mathbf{V}_K^H\text{diag}(\mathbf{d})\mathbf{V}_K\| \cdot \mathbf{1}_T^T\mathbf{U}_K^H\mathbf{U}_K\mathbf{1}_T$$

- Relates MSE with frequencies of control signals and node selection
- First term highlights the importance of selecting the frequency basis
- Second term reflects the impact of frequency content of control signals
- Role of statistics is explicit when controlling the mean system

Control Strategy

- Select subset $\mathcal{S}$ and design control signals $\mathbf{u}_t$ ⇒ Based on the statistics of the RES graph model
- Optimal strategy ⇒ Minimize the bound on the mean square error
- Selects precisely M nodes (fixed through time)
- Use δ to control the bias at time horizon T

Strategy

Determine $\mathcal{S} \subseteq \mathcal{V}$ through $\mathbf{d}$ and control signals with frequency content $\{\tilde{\mathbf{u}}_{t,K}\}_{t=0}^{T-1}$ such that

$$\begin{aligned} \underset{\substack{\mathbf{d} \in \{0,1\}^N \\ \mathbf{U}_K \in \mathbb{R}^{K \times T}}}{\text{minimize}} \quad & \|\mathbf{V}_K^H\text{diag}(\mathbf{d})\mathbf{V}_K\| \cdot \mathbf{1}_T^T\mathbf{U}_K^H\mathbf{U}_K\mathbf{1}_T \\ \text{subject to} \quad & \mathbf{d}^T\mathbf{1} = M, \\ & \|\mathbb{E}[\mathbf{x}_T] - \boldsymbol{\mu}_T\| \leq \delta. \end{aligned}$$

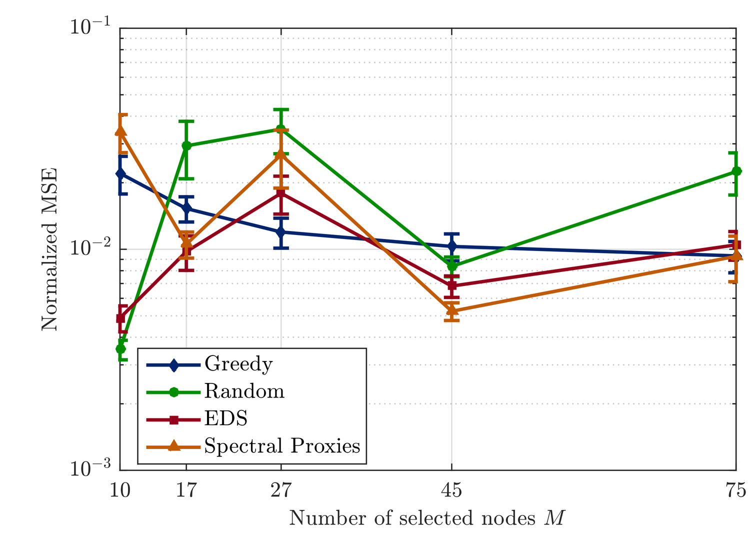
- **Not convex** on both $\mathbf{d}$ and $\mathbf{u}_t$ simultaneously
  - ⇒ Convex in each one of them, regarding the other as fixed
- Suboptimal approach ⇒ Select nodes so that system is controllable
  - ⇒ Then, optimize over $\mathbf{U}_K$ for selected nodes

Setup of Numerical Experiments

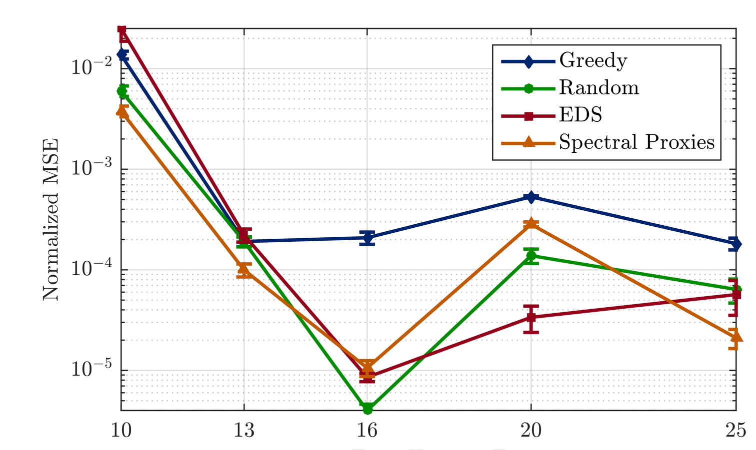
- Graph is a Stochastic Block Model (SBM) of $N = 300$ nodes
  - ⇒ Four communities, 75 nodes each
  - ⇒ Probability 0.9 of drawing edges within same community
  - ⇒ Probability 0.4 of drawing edges within different communities
- Drive signal to $\tilde{\boldsymbol{\mu}}_{T,K} = \mathbf{1}_K$ ⇒ Set $K = 10$
- Approaches for selecting node subset $\mathcal{S}$
  - ⇒ Greedy minimization of $\|\mathbf{V}_K^H\text{diag}(\mathbf{d})\mathbf{V}_K\|$
  - ⇒ Random node selection
  - ⇒ Select rows of $\mathbf{V}_K$ that maximize ∞-norm (EDS) [Varma '15]
  - ⇒ Spectral proxies (SP) method [Anis '16]
- Measure the normalized MSE (NMSE) w.r.t. mean control signal $\tilde{\boldsymbol{\mu}}_T$

Design Variables

- Fix edge activation probability $p_{\text{act}} = 0.9$ and either $T = 10$ or $M = 50$



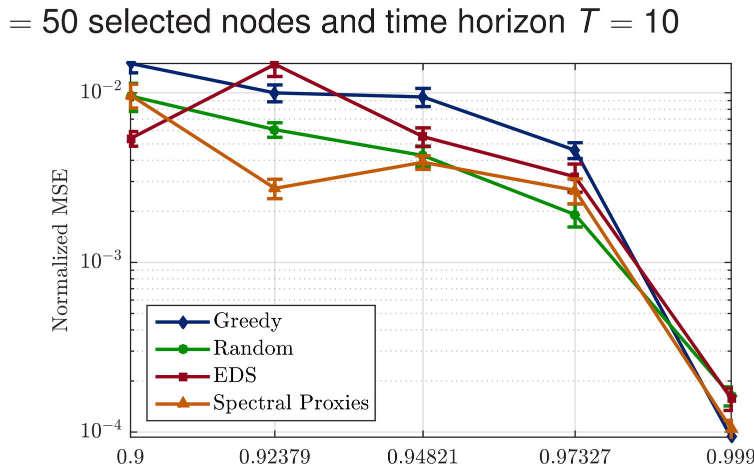
- NMSE drops as more nodes are controlled, especially for greedy



- NMSE drops as T increases ⇒ More time to control the signal

Random Graph Topology

- Fix $M = 50$ selected nodes and time horizon $T = 10$



- NMSE drops as $p_{\text{act}} \to 1$ ⇒ Control signal designed for mean graph

Conclusions and future work

- Controllability of graph signals diffused on random time-varying graphs
  - ⇒ Controllability in the mean ⇒ Drive signal w.r.t. expected graph
- Desired state is a bandlimited signal ⇒ Bandlimited control signals
- Fixed subset of nodes throughout the control process
- MSE analysis ⇒ Optimization problem for control strategy