Cramer-Rao bound for the Time-Varying Poisson
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Problems & results:
Periodic Time-varying Poisson new results:
1. the asymptotic Cramer-Rao bound,
2. a maximum likelihood parameter estimation.

Time-varying Poisson (tvP):
A sequence of random event times with independent increment, observed in period \([0, T]\).
Event times (ETs) \(T_n = \{T_1, \ldots, T_n\}\).
Counting process \(N_t = N(0, t] = \# \) of events up to and including time \(t\).
(deterministic) tvP intensity function:
\[ \lambda(t) = \lim_{\delta \to 0} \frac{1}{\delta} \Pr[N_{t+\delta} - N_t = 1], \]
Log-likelihood [1]:
\[ L_T = \int_0^T \ln \lambda(t) \, dN_t - \int_0^T \lambda(u) \, du. \]

Cramer-Rao bound (CRB) for tvP:
\(\theta\): parameter vector of dimension \(d\).
Theorem 1 tvP CRB [1]. For an unbiased estimator \(\hat{\theta}\),
\[ V_T = \var[\hat{\theta}] \geq I_T^{-1} \]
where \(I_T\) is the Fisher Information Matrix (FIM)
\[ I_T = \int_0^T \frac{\partial \lambda(t)}{\partial \theta} \frac{\partial \lambda(t)}{\partial \theta^T} \, dt, \]
and \(A \geq B\) means \(A - B\) is positive semi-definite.

Model assumption:
We assume periodic tvP intensity
\[ \lambda(t) = b + a \cos(\omega t), \]
So \(\theta = [b, a, \omega]^T\).

Averages of periodic functions:
• Suppose \(\gamma(t) \geq 0\) has period \(\frac{2\pi}{\omega}\).
• Then \(\kappa(t) = \gamma(t/\omega)\) has period \(2\pi\).
Result I Consider the integral
\[ J_T = \frac{1}{T} \int_0^T f(t) \, dt, \]
where \(f(t) = \sin^2(\omega t)\cos^2(\omega t)\) and \(p, q, r \geq 0\) are integers and \(|\alpha| < 1\). Then,
\[ J_T = \frac{1}{p + 1} \frac{1}{2\pi} \int_0^{2\pi} \kappa(\xi) \, d\xi. \]

Asymptotic CRB simplification:
• Introduce matrix \(D_T = \text{diag}(T^2, T^3, T^2)\).
Result II Asymptotic CRB.
\[ D_T V_T = I_T^{-1} \]
\[ = \left( D_T^{-1} I_T^{-1} D_T^{-1} \right)^{-1} \]
\[ = V_{\ast} \]
\[ G = \left[ \begin{array}{c} \frac{\rho}{R} \rho \end{array} \right] \]
\(R = 1 + \sqrt{1 - \rho^2}\).
Remarks.
\(\var[\hat{a}]\), \(\var[\hat{b}]\) decay at rate \(T^{-1}\) but
\(\var[\hat{\omega}]\) decays at \(T^{-3}\),
super efficiency of the frequency estimator.

MLE algorithm via cyclic ascent:
• First MLE for periodic tvP, surprisingly.
• Reparametrization: 2 positive components
\[ \lambda(t) = b + a \cos(\omega t) \]
\[ = c + a[1 + \cos(\omega t)] \]
• MLE via cyclic ascent:
  Stage I: update \(\hat{c}\) and \(\hat{a}\) by EM.
  Stage II: update \(\hat{\omega}\) by Newton-Raphson.
Formulae to be found in the paper.

Scale-free (SF) parameters and measure:
• SF parameters: \(n_b = bT, m_r = \omega T, \rho = \frac{a}{b}\).
• SF precision measure:
\[ \pi(\hat{\theta}) = \frac{m_T}{\sqrt{\pi}} \sqrt{\frac{m_T}{\sqrt{\pi} \sqrt{T}}} \]
where \(\sqrt{T} = \frac{\rho}{\sqrt{\pi}}\).
• Interpretations:
 1. Precision only increases linearly with \(\sqrt{T}\);
 2. But \(\hat{\omega}\) precision also rises linearly with \(m_T\):
greater precision;
 3. If \(\rho \approx 0\), precision for \(\hat{a}\) and \(\hat{\omega}\) very low.

Simulation setup:
• Parameters to be chosen: \(b, a, \omega, T\).
• One degree of freedom: fix \(\omega = 3\).
• We vary SF parameters on a \(8 \times 8 \times 8\) grid:
\(n_b = 15^2: 30^2\), \(m_T = 0.5: 2\), \(\rho = 0.3: 0.6\).
• Simulate 1,000 repeats on each grid point to estimate the precision:
\[ \tilde{\pi}(\hat{\theta}) = \text{mean}(\hat{\theta}) / \text{se}(\hat{\theta}) \]

Result: Heatmaps of \(\tilde{\pi}(\hat{\theta})\) and \(\hat{\theta}\) (128) Expect \(\tilde{\pi}(\hat{\theta}) \to 1\).