Succinct Data Structures for Small Clique-Width Graphs

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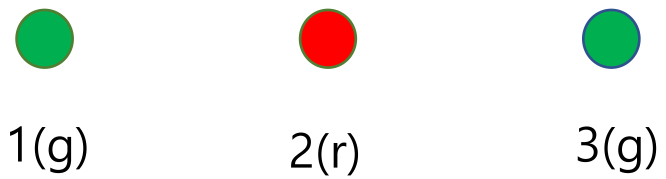
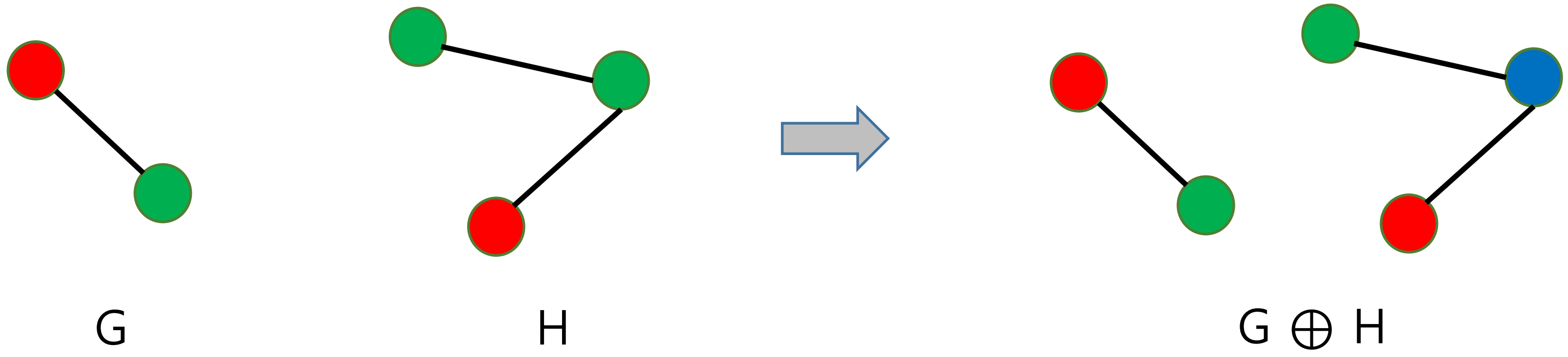
Clique-Width

Consider the following four operations on (undirected) graphs.

1. **Create** a vertex v with color i (denoted as v(i))



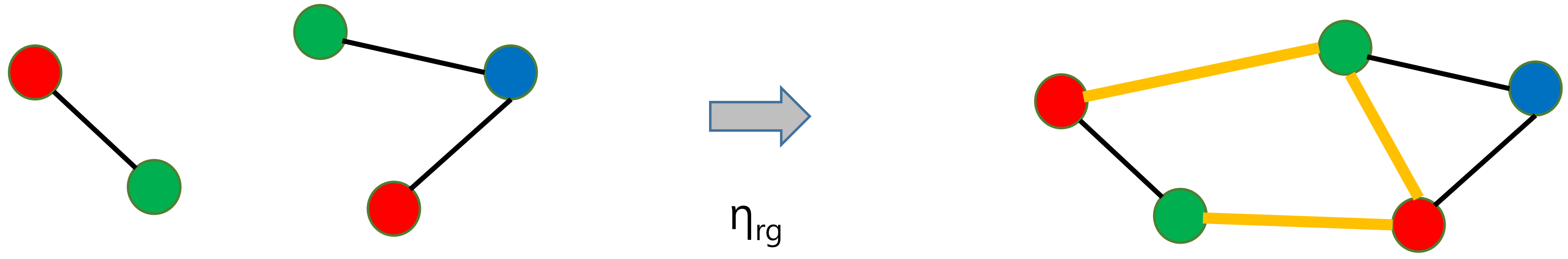
2. **Disjoint union** of labeled (colored) graph G and H (denoted as G ⨁ H)



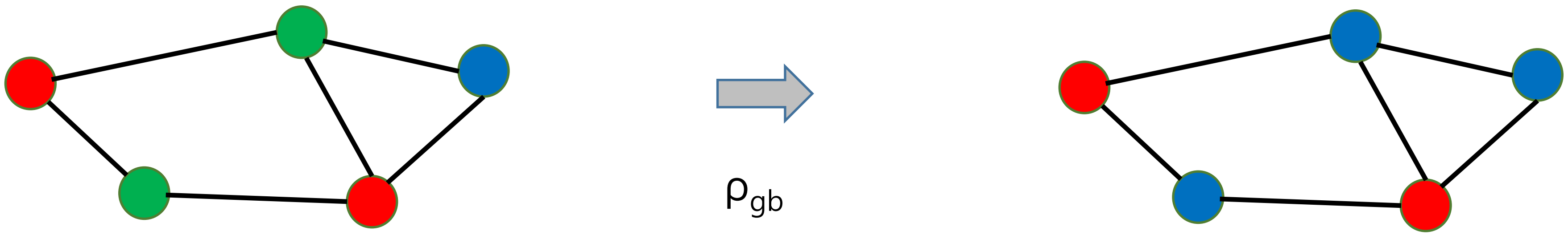
Clique-Width

Consider the following four operations on (undirected) graphs.

3. **Join** the color i and j (denoted as η_{ij})

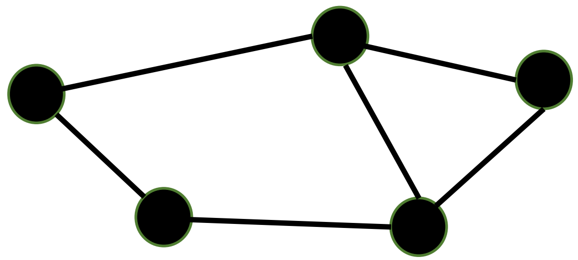


4. **Recolor** all the vertices of color i to j (denoted as ρ_{ij})



Clique-Width

- Clique-width of G (cw(G)) : **Number of minimum colors** to construct G using the previous four operations.



cw(G) = 3

Some examples

Clique-width 2 : Cliques, cographs….
Clique-width 3 : Distance-hereditary graph, 3-leaf power….

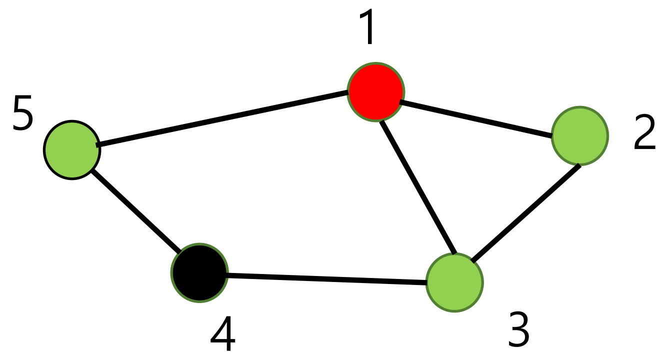
- Computing the clique-width of arbitrary graph is NP-hard.
- For small clique-width graphs, many NP-hard problems on general graphs (3-colorability, Hamiltonian cycles …) can be solved in polynomial time.

Problem statement

Problem : Given an undirected, unlabeled **graph G with n vertices whose clique-width is k**, is there any space-efficient data structure to support the following queries in efficient time ?

For any vertices u, v ∈ G

1. degree (v) : returns the degree of v.
2. neighborhood(v) : returns all the vertices adjacent to v.
3. adjacent (u, v) : returns true iff u and v are adjacent.



degree (1) = 3

neighborhood (1) = {2, 3, 5}

adjacent (1, 4) = false

Previous results & Our results

Problem : Given an undirected, unlabeled **graph G with n vertices whose clique-width is k**, is there any space-efficient data structure to support degree, neighborhood, and adjacent queries in efficient time ?

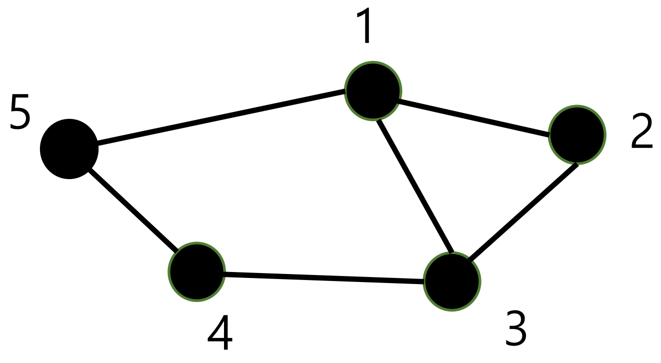
	Space (in bits)	degree	neighborhood (per neighbor)	adjacent
Kamali (2018)	O(kn) (O(knlog*n) bits for degree queries)	O(klog *n)	O(1)	O(1)
Our results ($k \le \epsilon\sqrt{\log \mathrm{n} / \log\log n}$)	f(n,k) + o(f(n,k))	O(k)	O(log n / k)	O(k)

- f(n, k): **Information-theoretical lower bound** of space to represent G.
- Kamali (2018) showed that (k-8)n ≤ f(n, k) ≤ 9kn, for k ≥ 9.
- Our data structure is **succinct** when $k \le \epsilon\sqrt{\log \mathrm{n} / \log\log n}$.
- For constant k, our data structure supports degree and adjacent query in O(1) time, and neighborhood query in O(log n) time per neighbor.

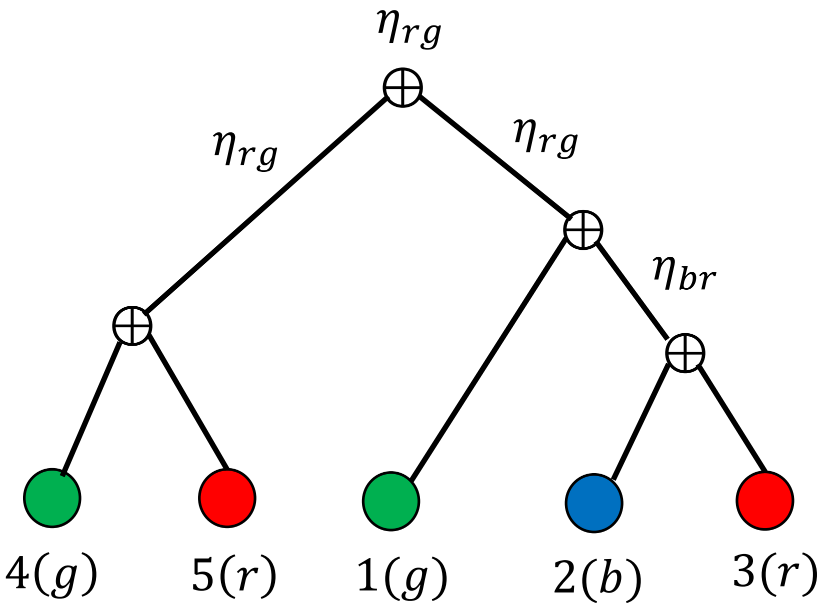
Succinct Encoding

- If one knows the clique-width of G (= k) there exists a **k-expression** that constructs G.
- Any k-expression can be represented by a labeled tree structure, named **union tree T**.
- One can reconstruct G from the union tree T of G

→ **Encoding of T gives an encoding of G !**



G (clique-width :3)



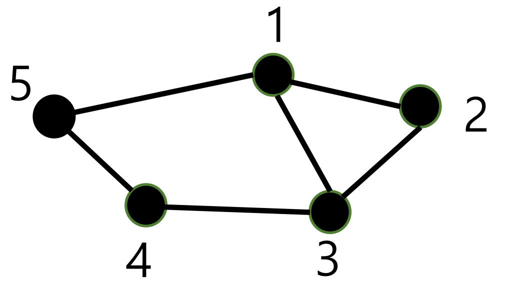
Union tree T of G

$\eta_{rg}(\eta_{rg}(4(g) \oplus 5(r)) \oplus \eta_{rg}, \eta_{br}((1(g) \oplus 2(b)) \oplus 3(r)))$ 3-expression of G

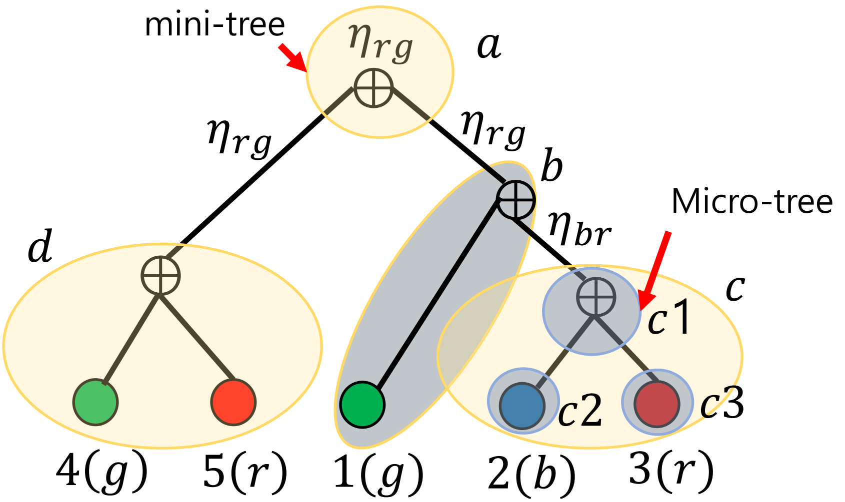
Succinct Encoding

Outline of the encoding :

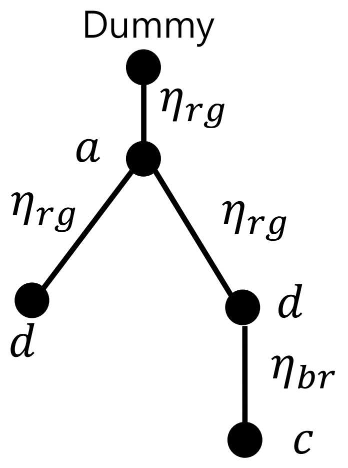
How to encode T ? : **Tree covering** [FM14]



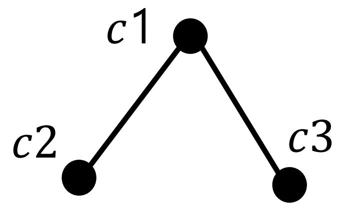
G



T



Tree over mini-tree



Mini-tree over micro-tree

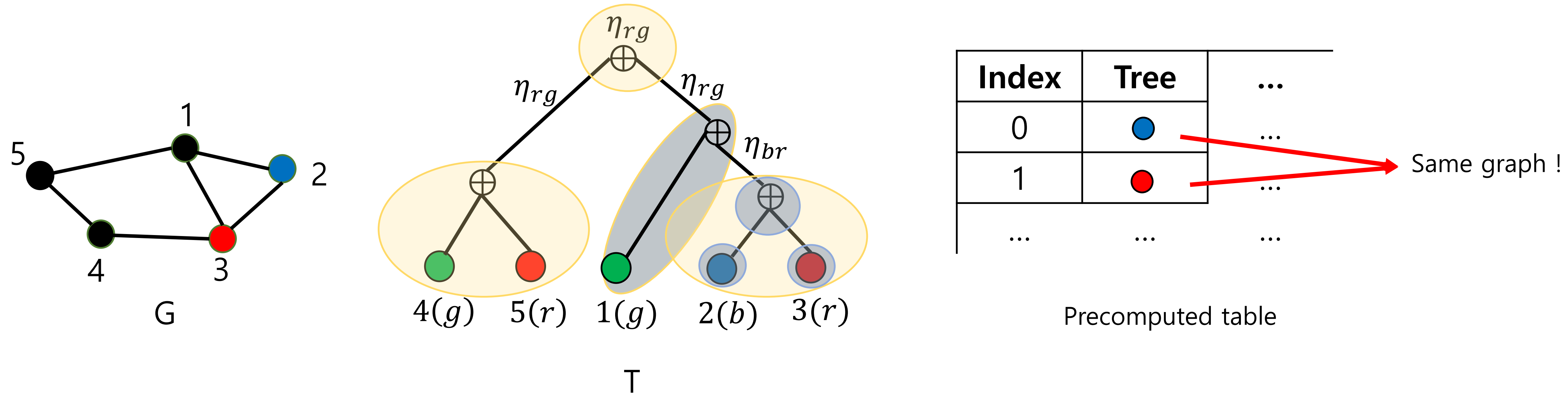
Index	**Tree**	**…**
0		…
1		…
…	…	…

Precomputed table

- Two-level decomposition algorithm (T → mini-tree → micro-tree)
- Tree over mini-trees, and mini-tree over micro-trees are stored using the pointer-based representation.
- Each micro-tree is stored as the **index of the precomputed table**, which stores all possible types of the micro-trees (with additional information for queries).
- Can support wide range of navigation queries on trees in O(1) time.

Succinct Encoding

Outline of the encoding



- Tree on micro-trees and mini-tree over micro trees can be stored in o(kn) bits of space.

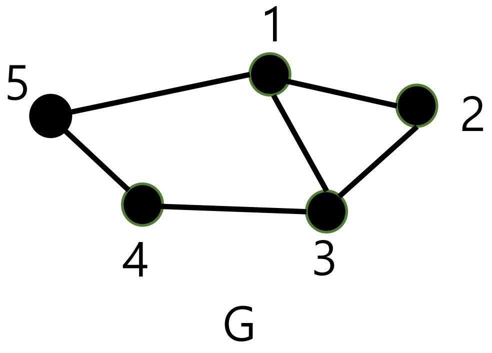
Problem : #non-isomorphic (colored, labeled) micro-trees >> #non-isomorphic clique-width k graphs with same size

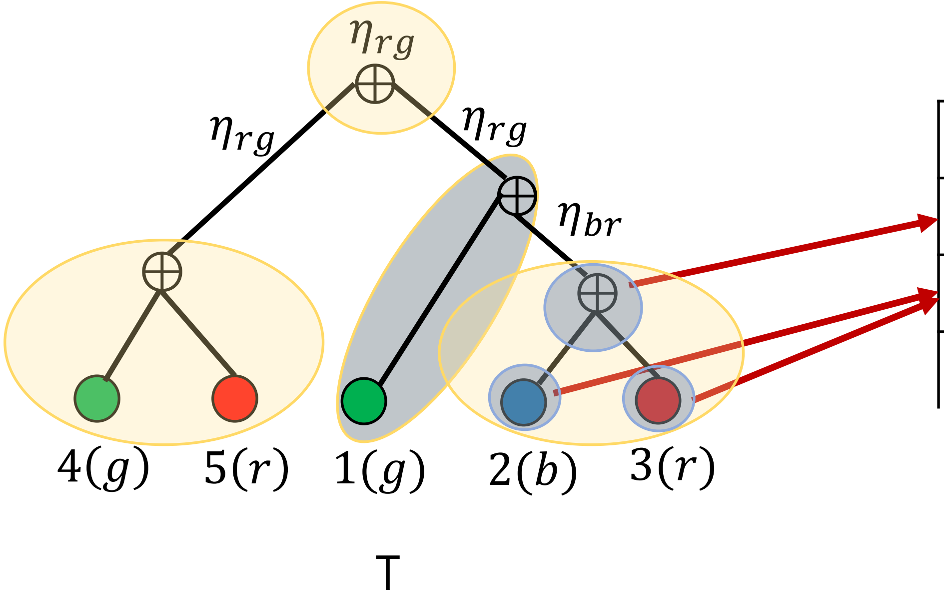
→ Size of the **pre-computed table is too large to encode every micro trees of T** using the index of the table in succinct space.

How to solve this problem ?

Succinct Encoding

Outline of the encoding





Index	Graph	...
0	Empty	...
1	●	...
...	...	...

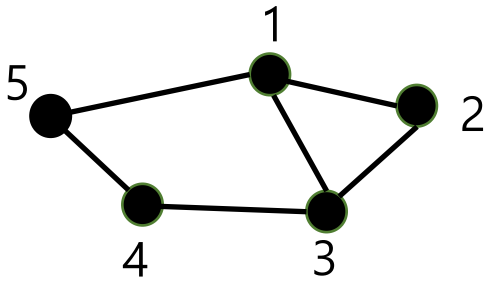
Precomputed table

Solution : Maintain a precomputed table to store every clique-width k graph (proportional to the size of the micro-tree of T), instead of the micro-tree.

Problem : Loose the information about the colors of vertices.

Succinct Encoding

Outline of the encoding



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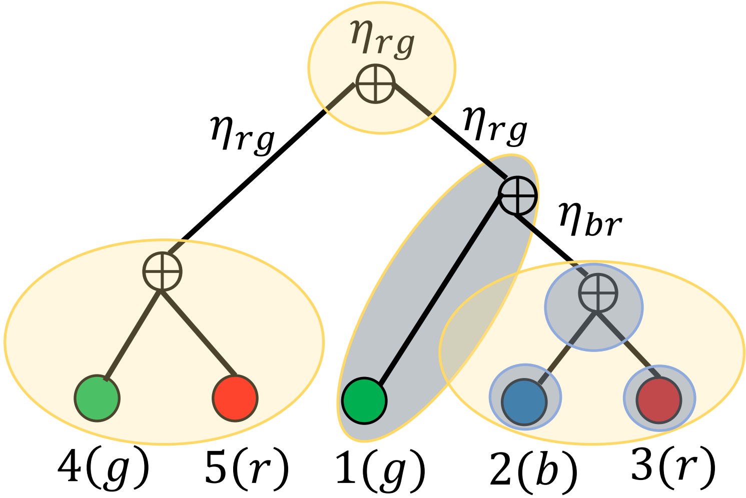


T

Index	Graph	...
0	Empty	...
1	●	...
...	...	...

Precomputed table

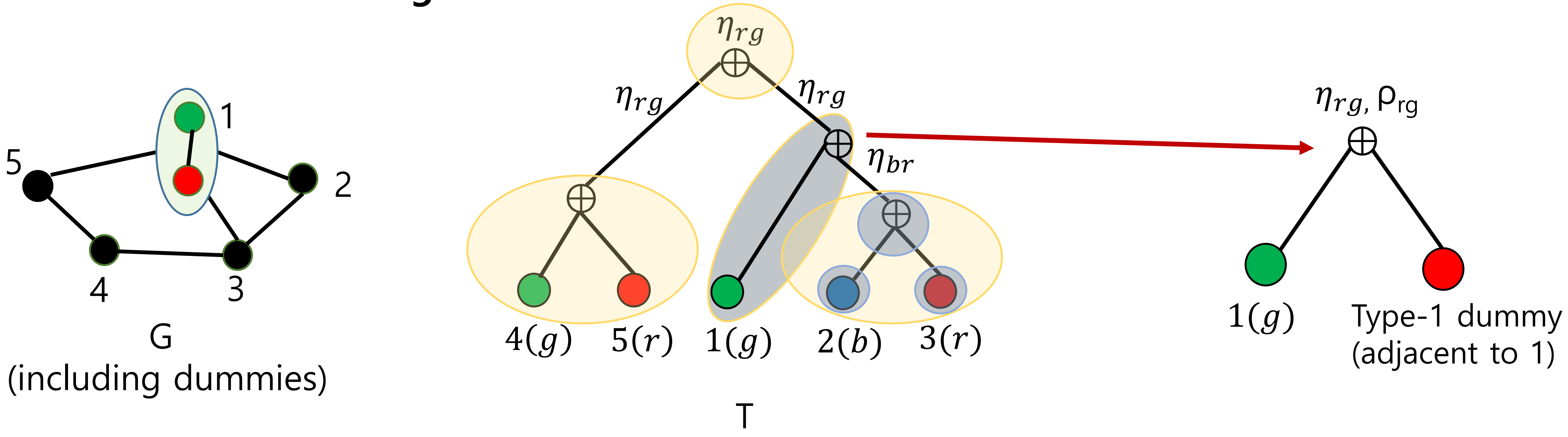
Problem : Loose the information about the colors of vertices.

Key observation : We only need color of vertices **at the root of the micro-tree.**

→ Add some dummy nodes to decode them.

Succinct Encoding

Outline of the encoding



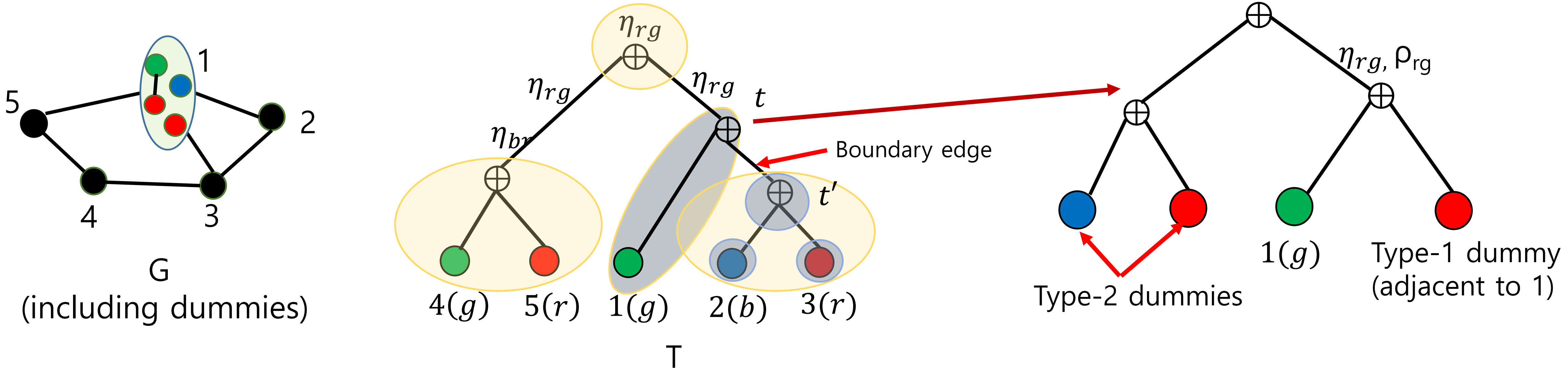
We define two types of dummy nodes on each micro-tree

Type-1 dummy nodes : To decode the color of each vertices at the root of the micro-tree.

→ Decode the color by checking the adjacency with the dummy nodes (using precomputed table).

Succinct Encoding

Outline of the encoding

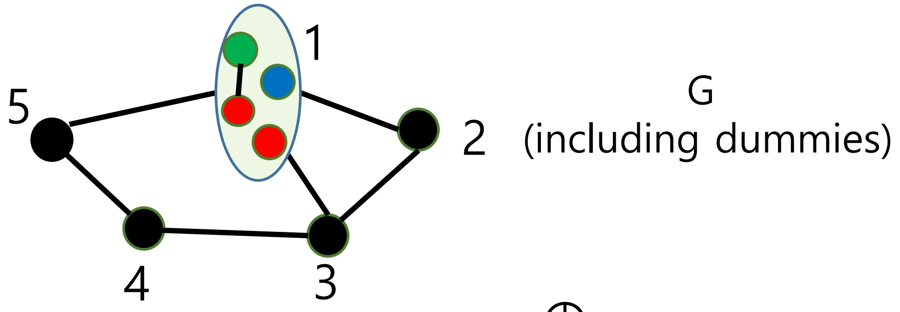


We define two types of dummy nodes on each micro-tree

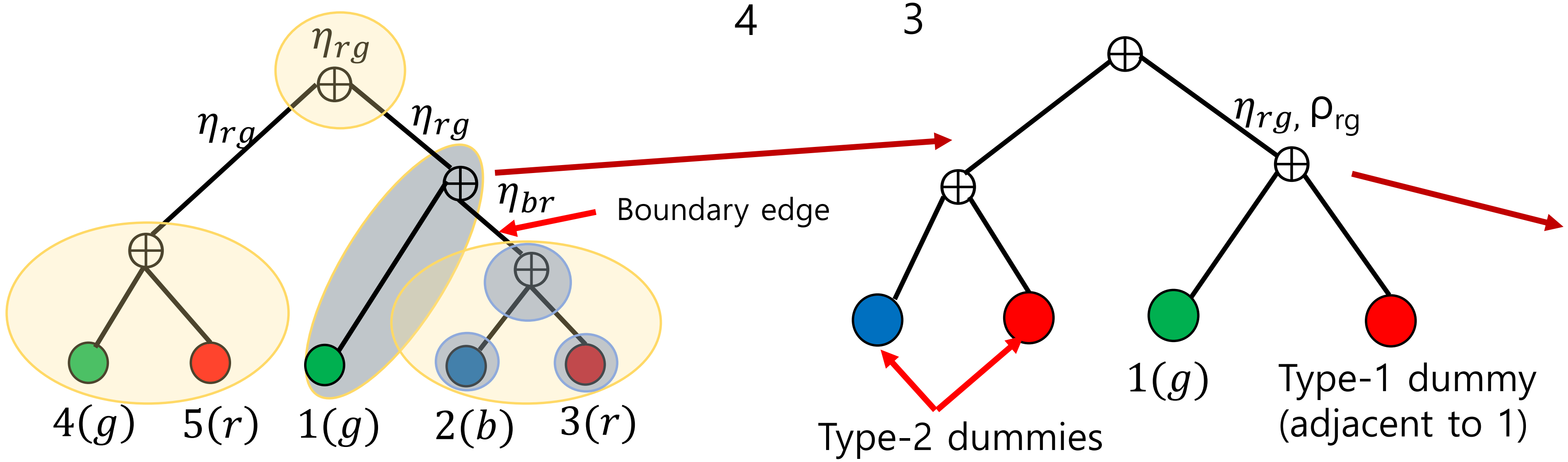
- Each micro tree t is connected by at most 1 another micro-tree t′ by a boundary edge.

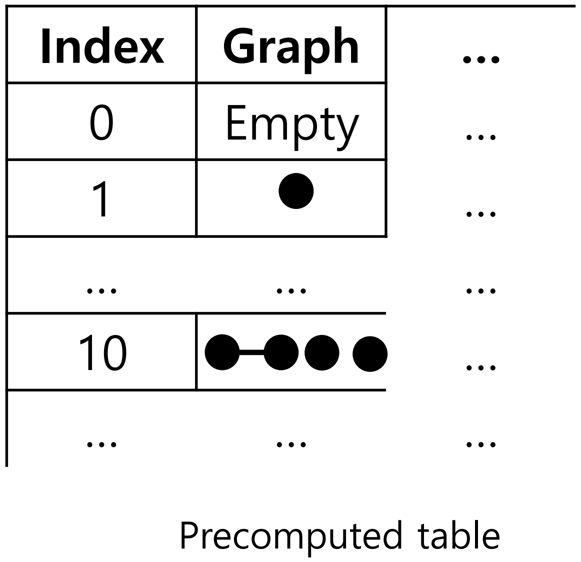
Type-2 dummy nodes : To decode the connection between the vertices in t and all the vertices in the subtree at the root of t′.

Succinct Encoding



Outline of the encoding





- We encode the corresponding graph of the micro-tree (with Type-1 and 2 dummy nodes) as an index of the precomputed table.

- The additional information of dummy nodes (position, colors….) is stored explicitly.

- Since the k is small ($k \leq \epsilon\sqrt{\log \mathrm{n}/\log\log n}$), and there exists at most 2k dummy nodes for each micro-tree of T, all the additional information can be stored in succinct space.

Query Algorithms

- Maintain the similar auxiliary structures of Kamali (2018), with some modifications for keeping the information on the nodes in tree over micro-trees.

- There exists some time blow-up for neighborhood queries, since we need to search every vertex in the micro-tree which has at least one neighborhood of the query vertex.

Conclusion

- Succinct data structure for the graphs with small bounded clique-width.

- Compare to the Kamali (2018)'s result, our data structure can support degree queries in O(k) time, still using succinct space.

- Since the cograph is equivalent to the graph with clique-width 2, our data structure gives a succinct data structure for cographs.

Further improvements (not in the paper)

- Succinct ds for cograph with O(1) time neighborhood query (per neighbor).
- Succinct ds for distance-hereditary graphs and Ptolemaic graphs (subclasses of the graph whose clique-width is 3).

Open question

: Currently succinct data structure for graphs with bounded width parameter is only considered for tree-width (FK14) and clique-width. Can we design succinct data structures w.r.t. other width parameters?