Rumour Source Detection in Social Networks using Partial Observations

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Problem Statement and Assumptions

Network topology
• General graph with small-world property.

Epidemic model
• Discrete-time version of susceptible-infected model.
• Constant transmission rate within the network.

Observation model
• Known graph topology.
• Monitoring of a small fraction of nodes.
Problem Statement and Assumptions

Source localisation problem

• A source emits $R$ rumours, at $t_0 = 0$.

• We observe some monitors, at discrete times $t \in \{0, 1, \ldots, T\}$.

• The probability of infection of a monitor $i$ at time $t$ is given by:

\[ \tilde{F}_i(t) = \frac{R_i(t)}{R}, \]

where $R_i(t)$ is the number of rumours which have reached $i$ by time $t$.

• We aim to leverage the divergence of the monitor measurements from an analytical probability of infection.
Approach I to Model Diffusion in a Network

What is the probability a node $i$ gets first infected at time $t$, $f_i(t)$?

$$f_i(t) = P(A \cap B) = P(A|B)P(B)$$

$\mu$ is the constant transmission rate

Derivation in spirit with the methods presented in:
Approach I to Model Diffusion in a Network

What is the probability a node $i$ gets first infected at time $t$, $f_i(t)$?

Event B

Event A

$t - 1$ $[t - 1, t]$ $t$

$B$ is the event of node $i$ being in a susceptible state at time $t - 1$:

$$P(B) = \prod_{\tau=1}^{t-1} (1 - f_i(\tau))$$
Approach I to Model Diffusion in a Network

\[ P(A) = 1 - \prod_{j \in N_i} [1 - \mu \times F(x_j(t - 1) = 1)] \]

- \( k \) is infected
- \( x_j(t - 1) = 1 \)
- neighbour \( j \) infected
- neighbour \( j \) does not transmit
- none of neighbours transmit

Event B

Event A

\( t - 1 \)
The probability \( i \) gets the rumour from at least one neighbour, given \( i \) was previously in a susceptible state is:

\[
P(A|B) = 1 - \prod_{j \in N_i} [1 - \mu \times F(x_j(t - 1) = 1|\mathbf{x}_i(t - 1) = 0)]
\]
Approach I to Model Diffusion in a Network

The probability a node $i$ gets first infected at time $t$, $f_i(t)$ is:

$$f_i(t) = [1 - \prod_{j \in N_i} (1 - \mu \times F(x_j(t-1) = 1|x_i(t-1) = 0))] \times \prod_{\tau=1}^{t-1} (1 - f_i(\tau))$$

where $N_i$ is the neighborhood of node $i$, $\mu$ is the infection rate, and $F$ is the infection function.
Approach I to Model Diffusion in a Network

The probability a node \( i \) gets first infected at time \( t \) is:

\[
f_i(t) = \left[ 1 - \prod_{j \in N_i} (1 - \mu \times F(x_j(t - 1) = 1 | x_i(t - 1) = 0) \right] \times \prod_{\tau=1}^{t-1} (1 - f_i(\tau))
\]

We make the approximation:

\[
f_i(t) \approx \left[ 1 - \prod_{j \in N_i} (1 - \mu \times F(x_j(t - 1) = 1) \right] \times \prod_{\tau=1}^{t-1} (1 - f_i(\tau))
\]

The approximate probability a node \( i \) is infected at time \( t \) is:

\[
F_i(\tau) \approx \sum_{t=1}^{\tau} \left[ 1 - \prod_{j \in N_i} (1 - \mu \times F(x_j(t - 1) = 1) \right] \times \prod_{\theta=1}^{t-1} (1 - f_i(\theta))
\]
Approach I to Model Diffusion in a Network

- The approximate probability a node $i$ is infected at time $t$ is:

\[
F_i(t) \approx \sum_{t=1}^{\tau} [1 - \prod_{j \in N_i} (1 - \mu \times F(x_j(t-1) = 1))] \times \prod_{\theta=1}^{t-1} (1 - f_i(\theta))
\]

- Spreading of 1000 Rumors, small-world network, 200 Nodes, for distances 1, 2, and 3 from the source:
Approach II to Model Diffusion in a Network

- Probability of infection based on the shortest distance to the source.
- Arrange the nodes according to the shortest distance to the destination.
- What is the probability of first infection of a node $i$ at distance $d$, at time $t$?
Approach II to Model Diffusion in a Network

- What is the probability of first infection of a node $i$ at distance $d$, at time $t$?
- Success: move closer to node $i$.

![Diagram showing diffusion process with different distances and times](image-url)
Approach II to Model Diffusion in a Network

- What is the probability of infection of a node $i$ at distance $d$, at time $t$?
- Failure: not spreading the rumour to a sufficient number of nodes closer to the destination.
Approach II to Model Diffusion in a Network

- Number of paths is the number of ways to choose \( d - 1 \) success steps of \( t - 1 \) time steps: \( \binom{t-1}{d-1} \).
- Probability of success: \( p_S \).
- Probability of each path: \( p_S^d \times (1 - p_S)^{t-d} \).
- Approximate \( p_S = \alpha_d \mu \), where \( \mu \) is the constant transmission rate in the graph.
Approach II to Model Diffusion in a Network

- Number of paths is: \( \binom{t-1}{d-1} \).
- Probability of each path: \( p_S^d \times (1 - p_S)^{t-d} \).
- Set \( p_S = \alpha_d \mu \), where \( \mu \) is the constant transmission rate in the graph.
- The probability of first infection is:
  \[
  f_d(t) = \underbrace{\left( \alpha_d \mu \right)^d \times (1 - \alpha_d \mu)^{t-d}}_{p_S} \times \underbrace{\binom{t-1}{d-1}}_{\# \text{ of paths from source to destination}}
  \]
- The probability of infection of a node at distance \( d \) from the source at time \( \tau \) is:
  \[
  F_d(\tau) \approx \sum_{t=d}^{\tau} \left( \alpha_d \mu \right)^d \times (1 - \alpha_d \mu)^{t-d} \times \binom{t-1}{d-1}
  \]
Approach II to Model Diffusion in a Network

- 1000 Rumors, small-world network, 200 Nodes:
Single Diffusion Source Detection Algorithm

- **Estimate the distances** between each monitor \(i\) and the potential source, by computing the dissimilarity between the observed \(\tilde{F}_i(t)\) and the theoretical \(F_d(t)\).

- Create a set of potential sources using **triangulation**.

- Select the most likely rumour origin, using the approximate model of infection, given a rumour source \(s\):

\[
F(x_i(\tau) = 1|s) \approx \sum_{t=1}^{\tau} \prod_{j \in N_i} (1 - \mu \times F(x_j(t-1) = 1)) \times \prod_{\theta=1}^{t-1} (1 - f_i(\theta))
\]

- For each potential source \(s\), compute the dissimilarity between empirical \(\tilde{F}_i(t)\) and analytical \(F(x_i(T) = 1|s)\). The **most likely rumour origin** is the node with the **lowest dissimilarity**.
Simulations

- 10 Rumors, small-world network, 1000 Nodes, $\mu = 0.5$, 100 experiments.
Simulations

- 10 Rumors, Facebook network, 192 Nodes, $\mu = 0.5$, 100 experiments.
Conclusion

• Mathematical models of information propagation, which accurately capture the diffusion process.

• Source detection algorithm, which assumes:
  – Single source, which emits multiple rumours.
  – All rumours start at the same time, which is known.
  – A finite set of monitor nodes is observed at discrete times.

• Future extensions:
  – Source detection with unknown start time.
  – Multiple source detection algorithm.
Thank you for listening!
How do we find the optimal parameters $\alpha_d$ in the distance-dependent probabilities?

- The distance-dependent probability of infection for a node at distance $d$, at time $t$ is:

  $$F_d(t) = \sum_{\tau=d}^{t} (\mu \times \alpha_d)^{d} \times (1 - \mu \times \alpha_d)^{\tau-d} \times \left(\frac{\tau - 1}{d - 1}\right)$$

- Artificially spread a number of rumours from a random node in the network, and obtain the empirical probabilities $\tilde{F}_i(t)$.

- The optimal parameter $\alpha_d$ minimizes the dissimilarity between $F_d(t)$ and $\tilde{F}_i(t)$ for a particular distance $d$:

  $$\alpha_d^{opt} = \arg\min_{\alpha_d} \sum_{i \in N_d} \sum_{t=0}^{T} ||F_d(t) - \tilde{F}_i(t)||^2,$$

  where $N_d$ is the set of nodes at shortest distance $d$ from the source.
How do we estimate the shortest distances between monitor nodes and the source?

- We find the dissimilarity between the distance-dependent analytical probability of infection $F_d(t)$, and the observed infection probability at a node $i$, using mean-squared error.

- Then, the optimal distance for a monitor $i$ is:

\[
d_{i,s} = \text{argmin}_d \sum_{t=0}^{T} ||F_d(t) - \tilde{F}_i(t)||^2
\]

- We select as potential sources all the nodes at distance $d_{i,s}$ from node $i$. 
How do we select the most likely rumour origin?

- Select the most likely rumour origin, using the approximate model of infection, given a rumour source $s$:

$$F(x_i(T) = 1|s) = \sum_{t=1}^{T} \left[ 1 - \prod_{j \in N_i} 1 - \mu \times F(x_j(t - 1) = 1) \right] \times \prod_{\tau=1}^{t-1} (1 - f_i(\tau))$$

- For each potential source $s$, compute the dissimilarity between the observed infection probabilities of all monitors, and the theoretic model of infection:

$$\bar{C}(s) = \sum_{i} \sum_{t=0}^{T} ||F(x_i(t) = 1|s) - \tilde{F}_i(t)||^2$$

- The most likely rumour origin is the node with the lowest dissimilarity.
More simulation results

- 10 Rumors, small-world network, 1000 Nodes, *varying* spreading probability
More simulation results

- 10 Rumors, Facebook network, 192 Nodes, *varying* spreading probability
Probability of infection

- A node has the infection at time $t$ if it got initially infected at any of the times before, $\tau = 1, 2, ..., t$.

- The events of a node getting the initial infection at different times are mutually disjoint.

- Hence, the probability of infection is given by the sum of the likelihoods of first infection at different discrete times:

$$F_i(t) = \sum_{\tau=1}^{t} f_i(\tau)$$
Probability of being susceptible

- A node is susceptible at time $t$ if it didn’t get infected at any of the times before, $\tau = 1, 2, \ldots, t$.
- The events of a node not getting the initial infection at different times are mutually disjoint.
- Hence:

\[ \bar{F}_i(t) = \prod_{\tau=1}^{T} 1 - f_i(\tau) \]
Distance-dependent probability of infection

- Number of paths is: \( \binom{t-1}{d-1} \).
- Probability of each path: \( p_S^d \times (1 - p_S)^{t-d} \).
- A node at distance \( d \) gets infected if any succession of \( d \) success steps, and \( t - d \) failure steps happens.
- Different successions of S and F events are mutually disjoint.
- Hence, the probability of first infection is:

\[
f_d(t) = (\alpha_d \mu)^d \times (1 - \alpha_d \mu)^{t-d} \times \binom{t-1}{d-1}
\]

\[\text{# of paths from source to destination}\]
Distance-dependent probability of infection

- A node at distance $d$ gets infected if any succession of $d$ success steps, and $t - d$ failure steps happens.
- There can be a success following a failure, at the next time step.
Comparison to existing methods

- The authors in [1] propose a Monte Carlo method for single source estimation, with unknown infection time. In a random geometric graph, the probability of the origin to be within the first 10% ranked nodes is around 0.5 when observing 5% of the network, increasing to 0.9 when observing the full network.

- In a small-world network, our method achieves correct detection probability of 0.75 when observing 5% of the network, and 1 when observing the full network. The number of rumours is 2, and the rumour start time is known.