Generalized Approximate Message Passing (GAMP) for One-Bit Compressed Sensing with AWGN

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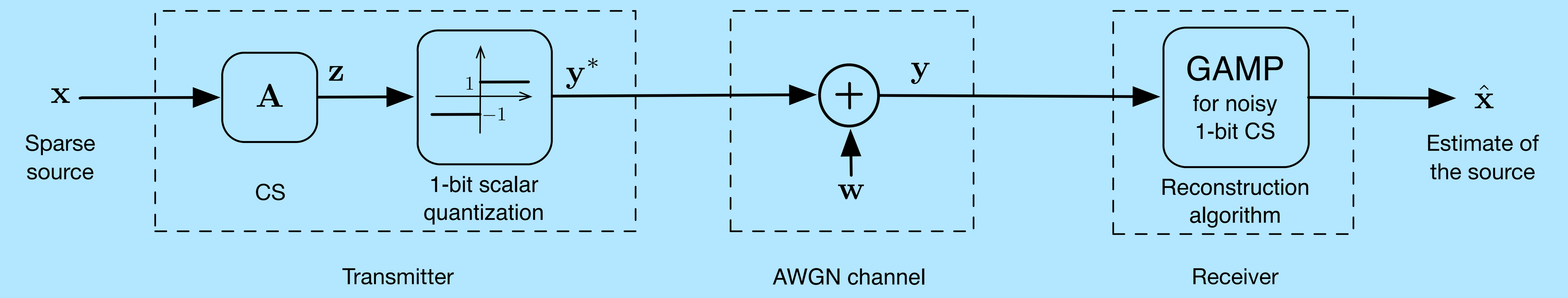


Introduction

Compressed Sensing

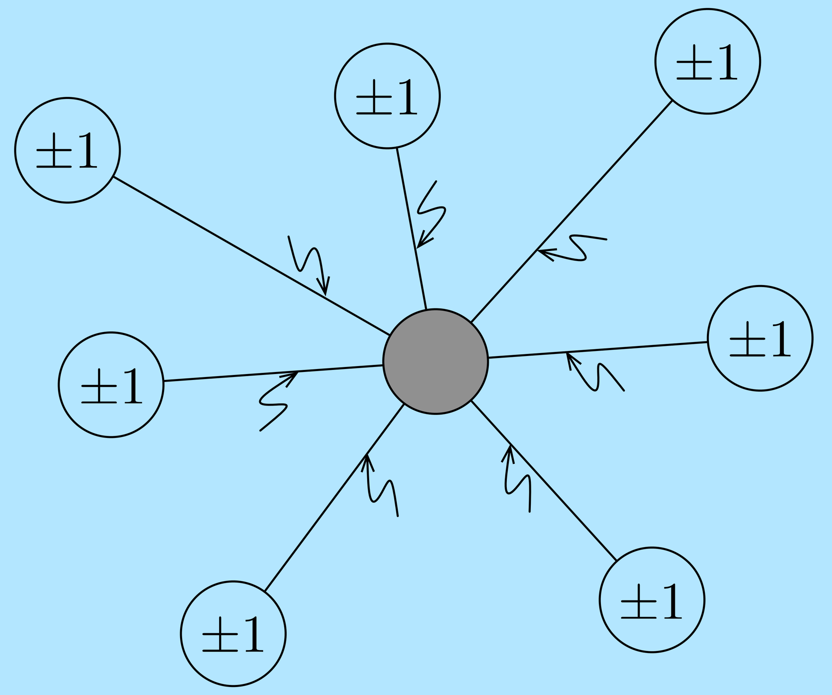
- In Compressed Sensing (CS) we take M linear measurements $\{y_i\}_{i=1}^{M}$ of an N-dimensional K-sparse (has at most K nonzero components) vector $\mathbf{x}$, according to

$$\mathbf{y} = \mathbf{A}\mathbf{x} \;\; (+\mathbf{w})$$

- Recovery is possible if $\mathbf{A} \in \mathbb{R}^{M\times N}$ satisfies the Restricted Isometry Property (RIP)
- Matrices whose elements are randomly drawn from a (sub-)Gaussian distribution satisfy RIP with high probability
- CS measurements need to be digitized for further processing

Basic Idea and Contributions

- Quantize CS measurement with 1-bit, send them through AWGN channel and apply efficient algorithm to recover $\mathbf{x}$ from noisy $\mathbf{y}$
- Application: scenarios with scarce bit budget
- The main contributions:
  - ▶ Consider a novel noise model for noisy 1-bit CS
  - ▶ Apply the known GAMP framework
  - ▶ Provide closed-form expressions for the nonlinear steps



1-bit CS measurements corrupted with AWG noise

Measurement Model

- We obtain M noisy 1-bit CS measurement according to

$$\mathbf{y} = \mathrm{sgn}(\mathbf{A}\mathbf{x}) + \mathbf{w}$$

- We assume i.i.d. noise vector $\mathbf{w}$ where each $w_i \sim \mathcal{N}(0, \sigma_w^2)$

Bernoulli-Gaussian Mixture Prior

We assume an i.i.d. source vector where each component x_j of $\mathbf{x}$ is a realization of a Bernoulli-Gaussian distributed random variable

$$p_{x_j}(x_j) = \gamma\delta(x_j) \; + \; (1-\gamma)\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2\sigma^2}x_j^2},$$

with γ being the probability of a zero-value and σ^2 being the variance of the zero-mean Gaussian distribution

GAMP for One-Bit Compressed Sensing with AWGN

Why GAMP?

- GAMP is very appealing for its efficiency and accurate reconstruction
- It approximates the computationally intractable high-dimensional integration involved with calculating

$$\hat{\mathbf{x}} \approx \mathbb{E}\{\mathbf{x} \mid \mathbf{y}\}$$

- It allows to model the quantization as a probabilistic channel with unquantized input and quantized output
- It allows to incorporate measurement noise in the model

The Steps of GAMP

- At $t = 0$, the algorithm is initialized according to (the far right values correspond to the Bernoulli-Gaussian mixture prior)

$$\hat{\mathbf{x}}^0 = \mathbb{E}\{\mathbf{x}\} = 0, \quad \mathbf{v}_x^0 = \mathrm{var}\{\mathbf{x}\} = (1-\gamma)\sigma^2, \quad \hat{\mathbf{s}}^0 = 0_{M\times 1}$$

- At every iteration $t = 1, 2, \ldots$ compute the measurement and estimation updates

$$\mathbf{v}_p^t = (\mathbf{A}\bullet\mathbf{A})\mathbf{v}_x^{t-1} \qquad \mathbf{v}_r^t = \left((\mathbf{A}\bullet\mathbf{A})^T\mathbf{v}_s^t\right)^{-1}$$
$$\hat{\mathbf{p}}^t = \mathbf{A}\hat{\mathbf{x}}^{t-1} - \mathbf{v}_p^t\bullet\hat{\mathbf{s}}^{t-1} \qquad \hat{\mathbf{r}}^t = \hat{\mathbf{x}}^{t-1} + \mathbf{v}_r^t\bullet(\mathbf{A}^T\hat{\mathbf{s}}^t)$$

$$\hat{\mathbf{s}}^t = \mathrm{F}_1(\mathbf{y}, \hat{\mathbf{p}}^t, \mathbf{v}_p^t) \qquad \hat{\mathbf{x}}^t = \mathrm{G}_1(\hat{\mathbf{r}}^t, \mathbf{v}_r^t; p_x)$$
$$\mathbf{v}_s^t = \mathrm{F}_2(\mathbf{y}, \hat{\mathbf{p}}^t, \mathbf{v}_p^t) \qquad \mathbf{v}_x^t = \mathrm{G}_2(\hat{\mathbf{r}}^t, \mathbf{v}_r^t; p_x)$$

- The functions $\mathrm{F}_1(\cdot)$, $\mathrm{F}_2(\cdot)$, $\mathrm{G}_1(\cdot)$ and $\mathrm{G}_2(\cdot)$ are applied component-wise and are given by

$$\mathrm{F}_1(y, \hat{p}, v_p) = \frac{\mathbb{E}\{z \mid y\} - \hat{p}}{v_p} \qquad \mathrm{G}_1(\hat{r}, v_r; p_x) = \mathbb{E}\{x \mid \hat{r}\}$$
$$\mathrm{F}_2(y, \hat{p}, v_p) = \frac{v_p - \mathrm{var}\{z \mid y\}}{v_p^2} \qquad \mathrm{G}_2(\hat{r}, v_r; p_x) = \mathrm{var}\{x \mid \hat{r}\}$$

- The first and the second moment of $z|y$ and $x|\hat{r}$ are evaluated with respect to

$$p_{z|y} \propto p_{y|z}(\cdot \mid \cdot)p_z(\cdot) \quad \text{and} \quad p_{x|\hat{r}} \propto g(\cdot; \hat{r}, v_r)p_x(\cdot)$$

where $z \sim \mathcal{N}(\hat{p}, v_p)$

- Stop iterating if $\|\hat{\mathbf{x}}^t - \hat{\mathbf{x}}^{t-1}\|_2 < \varepsilon\,\|\hat{\mathbf{x}}^t\|_2$ with a small $\varepsilon > 0$ (e.g. $\varepsilon = 10^{-2}$) or when $t \geq t_{\max}$

GAMP Operators for I/O Channels

- To compute the functions $\mathrm{F}_1(\cdot)$ and $\mathrm{F}_2(\cdot)$ we use the Bayes' rule to express conditional pdf as

$$f(z \mid y) = \frac{f(z)}{p(y)}p_w(y - \mathrm{sgn}(y)\mathrm{sgn}(z))$$

- The first and the second moment of $z \mid y$ are calculated as

$$\mathbb{E}\{z \mid y\} = C_y\,\hat{p} + (1 - C_y)\,z^*$$
$$\mathbb{E}\{z^2 \mid y\} = C_y\,(v_p + \hat{p}^2) + (1 - C_y)\,\tau_z^*$$

where C_y is a normalizing constant

- z^* and τ_z^* are the first and the second moment of $z \mid y^*$ whose calculation involves $\mathrm{erfcx}(x) = \mathrm{erfc}(x)\exp(x^2)$
- The functions $\mathrm{G}_1(\cdot)$ and $\mathrm{G}_2(\cdot)$ are calculated according to

$$\mathbb{E}\{x \mid \hat{r}\} = C^*\,\hat{r} \quad \text{and} \quad \mathbb{E}\{x^2 \mid \hat{r}\} = C^*\left(\frac{\hat{r}^2\sigma^2}{v_s} + v_r\right)$$

where C^* is a normalizing constant and $v_s = \sigma^2 + v_r$

Numerical Results

Simulation Setup

- We averaged our results over 1000 independent realizations of the source vector $\mathbf{x}$, the sensing matrix $\mathbf{A}$ and the AWGN $\mathbf{w}$
- In each simulation we acquire $M = 2000$ measurements of the underlying sparse vector of length $N = 512$
- Each 1-bit CS measurement vector is corrupted with AWG noise, where the SNR is defined as

$$\mathrm{SNR} = \mathbb{E}\{\|\mathbf{y}^*\|^2/\|\mathbf{w}\|_2^2\}$$

Results

- The gain in terms of MSE for SNRs below 2dB is about 5dB compared to modeling the noise with robustified activation function (GAMP with hard information)
- For high SNR this algorithm approaches the limit set by the GAMP algorithm for noiseless 1-bit CS measurements

Conclusion

- The GAMP algorithm with soft information outperforms (MSE-wise) the GAMP algorithm that uses hard information
- Exploiting soft information is possible with no additional cost