Compressed Sensing

Introduction

In Compressed Sensing (CS) we take M linear measurements \( (y)_{M} \) of an N-dimensional K-sparse (has at most K nonzero components) vector \( x \), according to

\[
y = Ax + w
\]

- Recovery is possible if \( A \in \mathbb{R}^{M \times N} \) satisfies the Restricted Isometry Property (RIP)
- Matrices whose elements are randomly drawn from a (sub-)Gaussian distribution satisfy RIP with high probability
- CS measurements need to be digitized for further processing

Basic Idea and Contributions

- Quantize CS measurement with 1-bit, send them through AWGN channel and apply efficient algorithm to recover \( x \) from noisy \( y \)
- Application: scenarios with scarce bit budget
- The main contributions:
  - Consider a novel noise model for noisy 1-bit CS
  - Apply the known GAMP framework
  - Provide closed-form expressions for the nonlinear steps

Compressed Sensing

1-bit CS measurements corrupted with AWGN noise

Measurement Model

- We obtain M noisy 1-bit CS measurement according to

\[
y = \text{sgn}(Ax) + w
\]

- We assume i.i.d. noise vector \( w \) where each \( w_i \sim \mathcal{N}(0, \sigma_t^2) \)

Bernoulli-Gaussian Mixture Prior

We assume an i.i.d. source vector where each component \( x_i \) of \( x \) is a realization of a Bernoulli-Gaussian distributed random variable

\[
p_{y}(x) = \gamma n(x_i) + (1-\gamma) e^{-\frac{x_i^2}{2\sigma_t^2}}
\]

with \( \gamma \) being the probability of a zero-value and \( \sigma_t^2 \) being the variance of the zero-mean Gaussian distribution

GAMP for One-Bit Compressed Sensing with AWGN

Why GAMP?

- GAMP is very appealing for its efficiency and accurate reconstruction
- It approximates the computationally intractable high-dimensional integration involved with calculating

\[
\hat{x} \approx \mathbb{E}[x | y]
\]

- It allows to model the quantization as a probabilistic channel
- It allows to incorporate measurement noise in the model

The Steps of GAMP

At \( t = 0 \), the algorithm is initialized according to (the far right values correspond to the Bernoulli-Gaussian mixture prior)

\[
\hat{x} = \mathbb{E}[x | y] \approx 0, \quad \psi = \var(x | \sigma) \approx (1-\gamma)\sigma^2, \quad \hat{p} = 0
\]

At every iteration \( t = 1, 2, \ldots \) compute the measurement and estimation updates

\[
\psi = (Ax + \hat{p})^{-1} \quad \hat{v} = (Ax + \hat{p})^{-1} \quad \hat{p} = A (Ax + \hat{p})^{-1} v
\]

- The functions \( F_1(\cdot) \), \( F_2(\cdot) \), \( G_1(\cdot) \) and \( G_2(\cdot) \) are applied component-wise and are given by

\[
F_1(y, \hat{p}, v) = \mathbb{E}[x | y] - \hat{p}
\]

\[
F_2(y, \hat{p}, v) = \mathbb{E}[x | y] - \hat{p}
\]

- The first and the second moment of \( z | y \) and \( x | p \) are evaluated with respect to

\[
p_{r} \propto p_{x}(r | \psi) p_{o}(\cdot) \quad \text{and} \quad p_{r} \propto g(\cdot; f, p, \psi)
\]

where \( z \sim \mathcal{N}(0, v) \)

- Stop iterating if \( \|x - \hat{x}\|_2 < \epsilon \|x\|_2 \) with a small \( \epsilon > 0 \) (e.g. \( \epsilon = 10^{-8} \)) or when \( t \geq \text{max} \)

GAMP Operators for I/O Channels

- To compute the functions \( F_1(\cdot) \) and \( F_2(\cdot) \) we use the Bayes’ rule to express conditional pdf as

\[
f(x | y) = \frac{f(x)}{f(y)} \frac{p_{x}(y | \psi)}{p_{y}(y | \psi)} \text{sgn}(x)
\]

- The first and the second moment of \( z | y \) are calculated as

\[
\mathbb{E}[x | y] = C_e \hat{p} + (1 - C_e) z^*
\]

\[
\mathbb{E}[x | y] = C_e (v + \hat{p}) + (1 - C_e) z^*_2
\]

where \( C_e \) is a normalizing constant

Compressed Sensing with AWGN

Simulation Setup

- We averaged our results over 1000 independent realizations of the source vector \( x \), the sensing matrix \( A \) and the AWGN \( w \)
- In each simulation we acquire \( M = 2000 \) measurements of the underlying sparse vector of length \( N = 512 \)
- Each 1-bit CS measurement vector is corrupted with AWGN noise, where the SNR is defined as

\[
\text{SNR} = \mathbb{E}[\|y\|_2^2] / \|w\|_2^2
\]

Results

- The gain in terms of MSE for SNRs below 2dB is about 5dB compared to modeling the noise with robustified activation function (GAMP with hard information)
- For high SNR this algorithm approaches the limit set by the GAMP algorithm for noiseless 1-bit CS measurements

Conclusion

- The GAMP algorithm with soft information outperforms (MSE-wise) the GAMP algorithm that uses hard information
- Exploiting soft information is possible with no additional cost