TONE RESERVATION AND SOLVABILITY CONCEPTS FOR THE PAPR PROBLEM IN GENERAL ORTHONORMAL TRANSMISSION SYSTEMS

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Introduction

- Large peak to average power ratios (PAPRs) can overload amplifiers, distort the signal, and lead to out-of-band radiation.
- The control of the PAPR is an important task in orthogonal waveform transmission schemes (e.g. orthogonal frequency division multiplexing (OFDM) and code division multiple access (CDMA)).
- There the PAPR can be as large as $\sqrt{\#\text{carriers}}$.
- The tone reservation method is an elegant and easy to define procedure to reduce the PAPR.
- We provide the first analytical result for PAPR reduction in general orthonormal systems.

PAPR

Peak to average power ratio (PAPR):
Ratio between the peak value and the square root of the power.

$$\mathrm{PAPR}(s) = \frac{\|s\|_{L^\infty[0,1]}}{\|s\|_{L^2[0,1]}}$$

(Note: usually the PAPR is defined as the square of this value.)

Orthogonal transmission scheme:

Transmit signal: $$s(t) = \sum_{k\in\mathcal{I}} c_k \phi_k(t), \quad t \in [0,1],$$

- $\{\phi_k\}_{k\in\mathcal{I}}$ is an orthonormal system (ONS) in $L^2[0,1]$.
- We assume that $\|\phi_k\|_\infty < \infty$, $k \in \mathcal{I}$ (bounded functions)
- Coefficients $c = \{c_k\}_{k\in\mathcal{I}} \subset \ell^2(\mathcal{I})$

PAPR: $$\mathrm{PAPR}(s) = \frac{\|\sum_{k\in\mathcal{I}} c_k\phi_k\|_{L^\infty[0,1]}}{\|c\|_{\ell^2(\mathcal{I})}}.$$

Large PAPRs are not specific to OFDM and CDMA systems.
→ They can occur for arbitrary bounded ONSs:

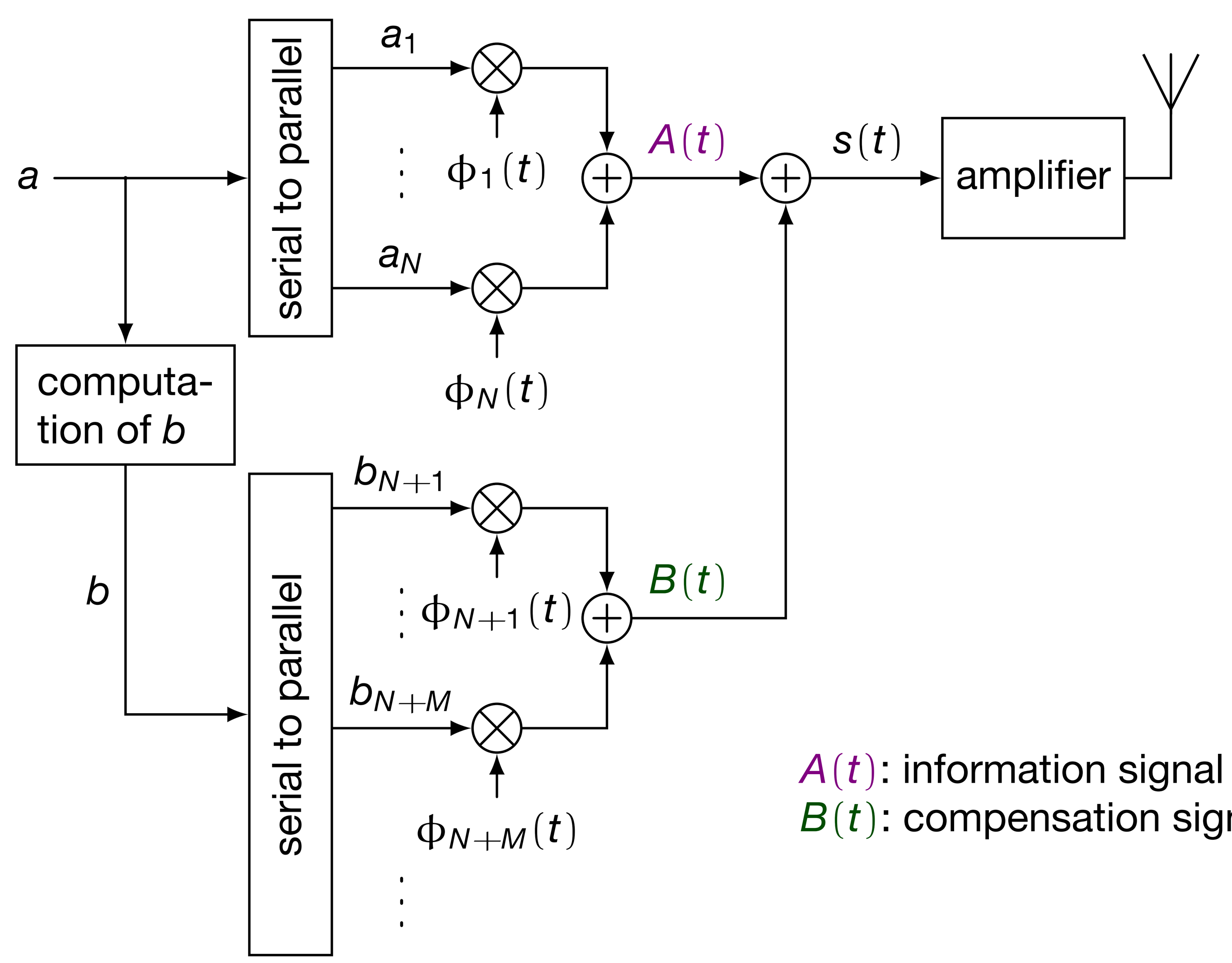
Example: Given any system $\{\phi_n\}_{n=1}^N$ of N orthonormal functions in $L^2[0,1]$, then there exist a sequence $\{c_n\}_{n=1}^N \subset \mathbb{C}$ of coefficients with $\sum_{n=1}^N |c_n|^2 = 1$, such that $\|\sum_{n=1}^N c_n\phi_n\|_{L^\infty[0,1]} \geqslant \sqrt{N}$.

Notation

W.l.o.g, we can consider signals defined on $[0,1]$.
$L^p[0,1]$, $1 \leqslant p \leqslant \infty$: the usual L^p-spaces on the interval $[0,1]$.
$\ell^2(\mathcal{I})$: set of all square summable sequences $c = \{c_k\}_{k\in\mathcal{I}}$ indexed by $\mathcal{I}$.
Norm: $\|c\|_{\ell^2(\mathcal{I})} = (\sum_{k\in\mathcal{I}} |c_k|^2)^{1/2}$.
Rademacher functions: $r_n(t) = \mathrm{sgn}[\sin(\pi 2^n t)]$.
Walsh functions: $w_1(t) = 1$ and $w_{2^k+m}(t) = r_{k+1}(t)w_m(t)$ for $k = 0, 1, 2, \ldots$ and $m = 1, 2, \ldots, 2^k$. Note: indexing starts with 1. The Walsh functions $\{w_n\}_{n\in\mathbb{N}}$ form an orthonormal basis for $L^2[0,1]$.

Tone Reservation Method

Orthogonal transmission scheme with tone reservation



$A(t)$: information signal
$B(t)$: compensation signal

Tone reservation method:
The index set $\mathcal{I}$ is partitioned in two disjoint sets $\mathcal{K}$ (information set) and $\mathcal{K}^\complement$ (compensation set). The set $\mathcal{K}$ is used to carry the information and the set $\mathcal{K}^\complement$ to reduce the PAPR.

For a given information sequence $a = \{a_k\}_{k\in\mathcal{K}} \in \ell^2(\mathcal{K})$, the goal is to find a compensation sequence $b = \{b_k\}_{k\in\mathcal{K}^\complement} \in \ell^2(\mathcal{K}^\complement)$ such that the peak value of the transmit signal

$$s(t) = \underbrace{\sum_{k\in\mathcal{K}} a_k\phi_k(t)}_{=:A(t)} + \underbrace{\sum_{k\in\mathcal{K}^\complement} b_k\phi_k(t)}_{=:B(t)}, \quad t \in [0,1],$$

is as small as possible.

$A(t)$: signal part which contains the information
$B(t)$: signal part which is used to reduce the PAPR

Strong Solvability and Compensation Sets

Example: For the Walsh ONS $\{w_n\}_{n\in\mathbb{N}}$ (CDMA case) we can use the information set $\mathcal{K} = \{2^l\}_{l\in\mathbb{N}\cup\{0\}}$. Then the PAPR problem is strongly solvable, and it can be shown that the optimal extension constant is $C_{\mathrm{EX}} = \sqrt{2}$ [BM18].
For the Fourier ONS $\{e^{ik\cdot 2\pi}\}_{k\in\mathbb{Z}}$ (OFDM case), the same information set $\mathcal{K} = \{2^l\}_{l\in\mathbb{N}\cup\{0\}}$ makes the PAPR problem strongly solvable. However, in this case the optimal extension constant is yet unknown.

[BM18] H. Boche and U. J. Mönich, "Optimal tone reservation for peak to average power control of CDMA systems," in *Proceedings of the IEEE International Conference on Acoustics, Speech, and Signal Processing (ICASSP '18)*, 2018, accepted

Strong and Weak Solvability

Definition (Strong solvability of the PAPR problem)

For an ONS $\{\phi_k\}_{k\in\mathcal{I}}$ and a set $\mathcal{K} \subset \mathcal{I}$, we say that the PAPR problem is strongly solvable with finite constant C_{EX}, if for all $a \in \ell^2(\mathcal{K})$ there exists a $b \in \ell^2(\mathcal{K}^\complement)$ such that

$$\Big\|\sum_{k\in\mathcal{K}} a_k\phi_k + \sum_{k\in\mathcal{K}^\complement} b_k\phi_k\Big\|_{L^\infty[0,1]} \leqslant C_{\mathrm{EX}}\|a\|_{\ell^2(\mathcal{K})}.$$

If the PAPR reduction problem is strongly solvable, we have:

- $\|b\|_{\ell^2(\mathcal{K}^\complement)} \leqslant C_{\mathrm{EX}}\|a\|_{\ell^2(\mathcal{K})}$
- $\mathrm{PAPR}(s) = \frac{\|s\|_{L^\infty(\mu)}}{\|s\|_{L^2(\mu)}} \leqslant \frac{C_{\mathrm{EX}}\|a\|_{L^2(\mu)}}{\|a\|_{L^2(\mu)}} \leqslant C_{\mathrm{EX}}$

Definition (Weak solvability of the PAPR problem)

For an ONS $\{\phi_k\}_{k\in\mathcal{I}}$ and a set $\mathcal{K} \subset \mathcal{I}$, we say that the PAPR problem is weakly solvable if for all $a \in \ell^2(\mathcal{K})$ we have

$$\inf_{b\in\ell^2(\mathcal{K}^\complement)} \Big\|\sum_{k\in\mathcal{K}} a_k\phi_k + \sum_{k\in\mathcal{K}^\complement} b_k\phi_k\Big\|_{L^\infty[0,1]} < \infty.$$

- The peak value of the transmit signal is only required to be bounded (and not to be controlled by the norm of the sequence $a = \{a_k\}_{k\in\mathcal{K}}$).
- Strong solvability always implies weak solvability.

If the PAPR reduction problem is weakly solvable, we have:

- $\|b\|_{\ell^2(\mathcal{K}^\complement)} \leqslant \|s\|_{L^\infty[0,1]} < \infty$

The equivalence "strong ⇔ weak" for OFDM was proved in [BMT17].

[BMT17] H. Boche, U. J. Mönich, and E. Tampubolon, "Complete characterization of the solvability of PAPR reduction for OFDM by tone reservation," in *Proceedings of the 2017 IEEE International Symposium on Information Theory*, Jun. 2017, pp. 2023–2027

Equivalence of Solvability Concepts

For arbitrary complete ONS, weak solvability implies strong solvability.
→ Both concepts of stability are equivalent.

Theorem: *Let $\{\phi_n\}_{n\in\mathbb{N}}$ be a complete ONS with $\sup_{n\in\mathbb{N}}\|\phi_n\|_\infty < \infty$, and $\mathcal{K} \subset \mathbb{N}$, such that the PAPR problem is weakly solvable. Then the PAPR problem is strongly solvable.*

Reduced Compensation Set

What happens if only a subset of $\mathcal{K}^\complement$ is used as compensation set?
The general answer is unknown.

Special case (OFDM, positive frequencies):
$I = \mathbb{N}$. The set $\{\phi_k\}_{k\in\mathbb{N}} = \{e^{ikt}\}_{k\in\mathbb{N}}$ is not complete in $L^2[-\pi,\pi]$.

Theorem: *Let $\mathcal{K} \subset \mathbb{N}$. The OFDM PAPR problem with reduced compensation set is strongly solvable if and only if it is strongly solvable with full compensation set.*