OPTIMAL TONE RESERVATION FOR PEAK TO AVERAGE POWER CONTROL OF CDMA SYSTEMS

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Introduction

- Large peak to average power ratios (PAPRs) can overload amplifiers, distort the signal, and lead to out-of-band radiation.
- The control of the PAPR is an important task in orthogonal waveform transmission schemes (e.g. orthogonal frequency division multiplexing (OFDM) and code division multiple access (CDMA)).
- There the PAPR can be as large as $\sqrt{\#\text{carriers}}$.
- The tone reservation method is an elegant and easy to define procedure to reduce the PAPR.
- We study the tone reservation technique for code division multiple access (CDMA) systems that employ the Walsh functions.

PAPR

Peak to average power ratio (PAPR):
Ratio between the peak value and the square root of the power.

$$\mathrm{PAPR}(s) = \frac{\|s\|_{L^\infty[0,1]}}{\|s\|_{L^2[0,1]}}$$

(Note: usually the PAPR is defined as the square of this value.)

Orthogonal transmission scheme:

Transmit signal: $s(t) = \sum_{k\in\mathcal{I}} c_k\phi_k(t), \quad t\in[0,1],$

- $\{\phi_k\}_{k\in\mathcal{I}}$ is an orthonormal system (ONS) in $L^2[0,1]$
- We assume that $\|\phi_k\|_\infty < \infty$, $k\in\mathcal{I}$ (bounded functions)
- Coefficients $c = \{c_k\}_{k\in\mathcal{I}} \subset \ell^2(\mathcal{I})$

PAPR: $\mathrm{PAPR}(s) = \dfrac{\|\sum_{k\in\mathcal{I}} c_k\phi_k\|_{L^\infty[0,1]}}{\|c\|_{\ell^2(\mathcal{I})}}.$

Large PAPRs are not specific to OFDM and CDMA systems.
→ They can occur for arbitrary bounded ONSs:

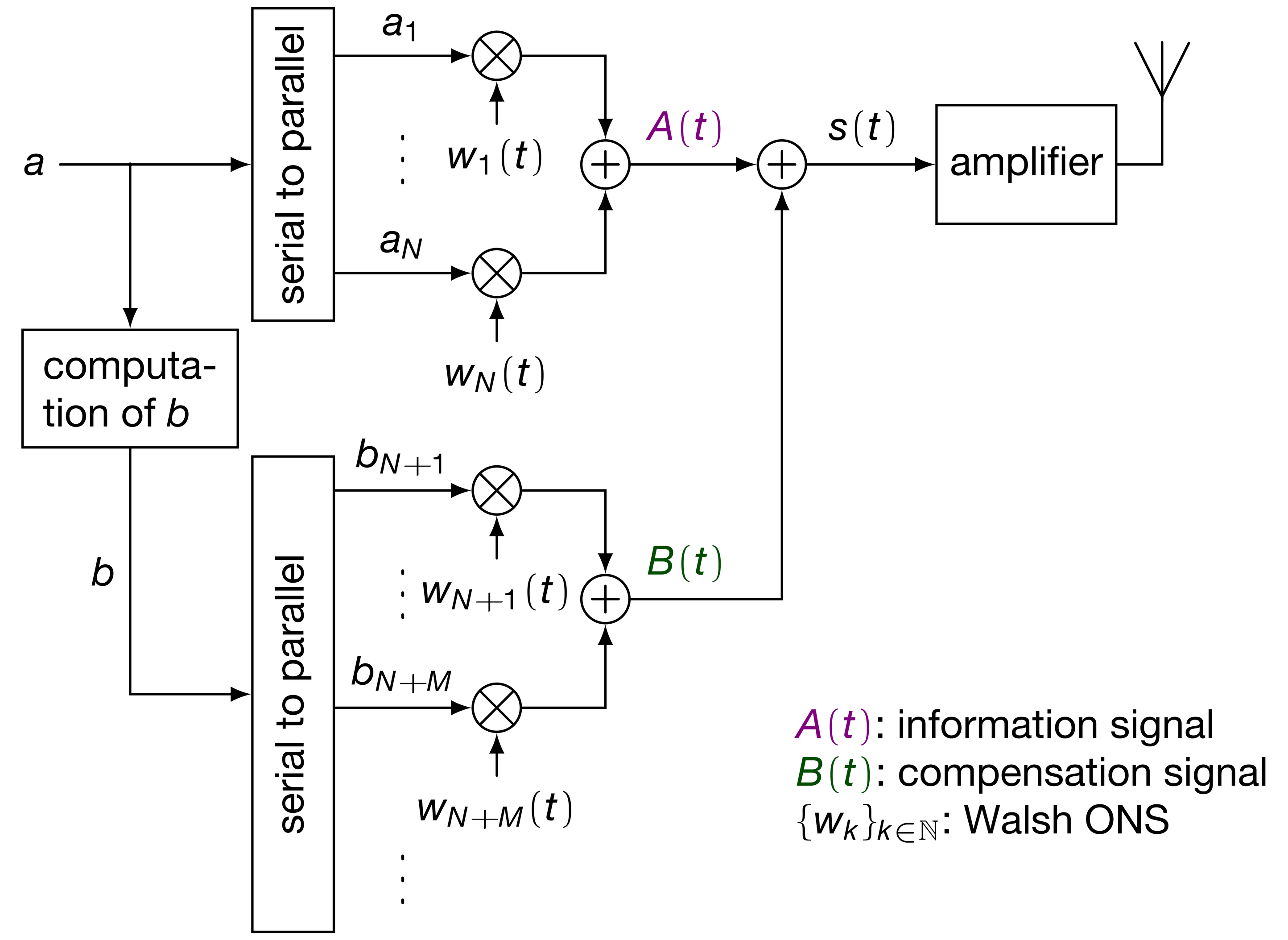
Example: Given any system $\{\phi_n\}_{n=1}^N$ of N orthonormal functions in $L^2[0,1]$, then there exist a sequence $\{c_n\}_{n=1}^N \subset \mathbb{C}$ of coefficients with $\sum_{n=1}^N |c_n|^2 = 1$, such that $\|\sum_{n=1}^N c_n\phi_n\|_{L^\infty[0,1]} \geqslant \sqrt{N}$.

Notation

$L^p[0,1]$, $1\leqslant p\leqslant\infty$: usual L^p-spaces on the interval $[0,1]$.
$\ell^2(\mathcal{I})$: set of all square summable sequences $c=\{c_k\}_{k\in\mathcal{I}}$ indexed by $\mathcal{I}$.
Norm: $\|c\|_{\ell^2(\mathcal{I})} = (\sum_{k\in\mathcal{I}}|c_k|^2)^{1/2}$.
Rademacher functions: $r_n(t) = \mathrm{sgn}[\sin(\pi 2^n t)]$.
Walsh functions: $w_1(t)=1$ and $w_{2^k+m}(t) = r_{k+1}(t)w_m(t)$ for $k=0,1,2,\ldots$ and $m=1,2,\ldots,2^k$. Note: indexing starts with 1. The Walsh functions $\{w_n\}_{n\in\mathbb{N}}$ form an orthonormal basis for $L^2[0,1]$.

Tone Reservation Method

CDMA transmission scheme with tone reservation



Tone reservation method:
The index set $\mathcal{I}$ is partitioned in two disjoint sets $\mathcal{K}$ (information set) and $\mathcal{K}^\complement$ (compensation set). The set $\mathcal{K}$ is used to carry the information and the set $\mathcal{K}^\complement$ to reduce the PAPR.

For a given information sequence $a=\{a_k\}_{k\in\mathcal{K}}\in\ell^2(\mathcal{K})$, the goal is to find a compensation sequence $b=\{b_k\}_{k\in\mathcal{K}^\complement}\in\ell^2(\mathcal{K}^\complement)$ such that the peak value of the transmit signal

$$s(t) = \underbrace{\sum_{k\in\mathcal{K}} a_k w_k(t)}_{=:A(t)} + \underbrace{\sum_{k\in\mathcal{K}^\complement} b_k w_k(t)}_{=:B(t)}, \quad t\in[0,1],$$

is as small as possible.

Solvability

Definition (Solvability of the PAPR problem for the Walsh ONS)
For an ONS $\{\phi_k\}_{k\in\mathcal{I}}$ and a set $\mathcal{K}\subset\mathcal{I}$, we say that the PAPR problem is solvable with finite constant C_{EX}, if for all $a\in\ell^2(\mathcal{K})$ there exists a $b\in\ell^2(\mathcal{K}^\complement)$ such that

$$\left\|\sum_{k\in\mathcal{K}} a_k w_k + \sum_{k\in\mathcal{K}^\complement} b_k w_k\right\|_{L^\infty[0,1]} \leqslant C_{\mathrm{EX}}\|a\|_{\ell^2(\mathcal{K})}.$$

If the PAPR reduction problem is strongly solvable, we have:

- $\|b\|_{\ell^2(\mathcal{K}^\complement)} \leqslant C_{\mathrm{EX}}\|a\|_{\ell^2(\mathcal{K})}$
- $\mathrm{PAPR}(s) \leqslant C_{\mathrm{EX}}$

Finding the optimal, i.e., minimal extension constant is an important problem that is relevant for applications.

Central Questions

1. What is the best possible reduction of the PAPR?
2. What is the optimal information set $\mathcal{K}$ that achieves this reduction, and how can it be found?
3. What is the general structure of the optimal information set $\mathcal{K}$?

Smallest Extension Constant

Let $\mathcal{K} = \{k_1,k_2,\ldots,k_N\}\subset\mathbb{N}$. By $C_{\mathrm{EX}}(\mathcal{K})$ we denote the optimal (smallest) extension constant for which the PAPR problem is solvable for the Walsh system $\{\phi_n\}_{n\in\mathbb{N}} = \{w_n\}_{n\in\mathbb{N}}$ and the set $\mathcal{K}$.

How small can the optimal extension constant become for different sets $\mathcal{K}$ of cardinality N?

$$\underline{C}_{\mathrm{EX}}(N) := \inf_{\substack{\mathcal{K}\subset\mathbb{N}\\ |\mathcal{K}|=N}} C_{\mathrm{EX}}(\mathcal{K}) \qquad (*)$$

Complete description of the smallest possible extension constant $\underline{C}_{\mathrm{EX}}$ (answer to Question 1):

Theorem: *We have $\underline{C}_{EX}(1)=1$ and $\underline{C}_{EX}(N)=\sqrt{2}$ for all $N\geqslant 2$.*

Optimal information set $\mathcal{K}^{\mathrm{opt}}(N)$ that achieves the best possible PAPR reduction (answer to Question 2):

Theorem: *For $N\in\mathbb{N}$ we have $\underline{C}_{EX}(N) = C_{EX}(\{2^k+1\}_{k=0}^{N-1})$. That is, $\mathcal{K}^{\mathrm{opt}}(N) = \{2^k+1\}_{k=0}^{N-1}$, showing that the first N Rademacher functions achieve the minimal extension constant $\underline{C}_{EX}(N)$.*

For each $N\in\mathbb{N}$ there indeed exists a set $\mathcal{K}^{\mathrm{opt}}(N)\subset\mathbb{N}$ with $|\mathcal{K}^{\mathrm{opt}}(N)|=N$, such that $\underline{C}_{\mathrm{EX}}(N) = C_{\mathrm{EX}}(\{\mathcal{K}^{\mathrm{opt}}(N)\})$. That is, the infimum in (*) is attained and in fact a minimum.

Structure of the Optimal Information Sets

The information sets $\mathcal{K}$ for which the PAPR is strongly solvable need to be sparse: If $\mathcal{K}\subset\mathbb{N}$ is a set such that the PAPR problem is solvable then we have

$$\lim_{N\to\infty}\frac{|\mathcal{K}\cap[1,N]|}{N} = 0.$$

Set of all optimal information sets $\mathcal{K}$:

$$T_N := \{\mathcal{K}\subset\mathbb{N}\colon |\mathcal{K}|=N, \underline{C}_{\mathrm{EX}}(N) = C_{\mathrm{EX}}(\mathcal{K})\}$$

We cannot conclude for $\mathcal{K}\in T_N$ and $k_l\notin\mathcal{K}$ that $\mathcal{K}\cup\{k_l\}\in T_{N+1}$.
Does there exists an infinite set $\mathcal{K}=\{k_1,k_2,\ldots\}$ such that the first N elements $K_N=\{k_1,\ldots,k_N\}$ always satisfy $K_N\in T_N$? The following corollary gives a positive answer.

Corollary: *Let $N\geqslant 2$ and $\mathcal{K}=\{k_1,\ldots,k_N\}\in T_N$. Then we have $\mathcal{K}\setminus\{k_l\}\in T_{N-1}$ for all $1\leqslant l\leqslant N-1$.*