NONLINEAR STATE ESTIMATION USING PARTICLE FILTERS
ON THE STEIFEL MANIFOLD
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1. Introduction
• Many engineering applications, such as attitude estimation, image processing, robotics, lead to models whose states are constrained to the Steifel manifold \( V_q \).
• This work extends [1] in two ways:
  – The observations are nonlinear functions of the state.
  – We approximate the optimal importance function.

2. Problem Setup
• Let \( S_t \) denote the state of a system on the Steifel manifold \( V_q \), i.e., \( V \in \mathbb{R}^{q \times q}, V^T V = I_q, \lambda > 1 \), according to:
  \[ S_{t+1} \sim \text{MVP}(S_t) \]
where \( \xi \in \mathbb{R}^k \) is a fixed hyperparameter and \( \rho(x) \) is the hypergeometric function with matrix argument.
• \( \{ S_t \} \) gives rise to the observation sequence \( \{ Y_t \} \), \( Y_t \in \mathbb{R}^{q \times q} \),
  \[ Y_t = g(S_t) + G_t \]
where \( g : \mathbb{R}^{q \times q} \rightarrow \mathbb{R}^d \) is a possibly nonlinear function, and \( N_q \) is a matrix normal distribution on \( \mathbb{R}^{q \times q} \).
• The particle filtering algorithm of [1] was restricted to \( G(S_t) = S_t \) and used the prior importance function:
  \[ p(S_t | S_{t-1}) \sim \text{MVP}(S_t | S_{t-1}) \]
The restriction on \( G \) can be trivially lifted, leading to the weight update equation:
  \[ w_{t+1} \propto w_t \rho(Y_t | g(S_t)) \]

3. Proposed Method
• Optimal importance function:
  \[ p(S_t | Y_t, S_t^{[0]}) \approx p(S_t | Y_t, S_t^{[0]}) \]
• The integral in (1) cannot be analytically evaluated if \( G(S_t) \) is a general nonlinear function. By linearizing \( G(S_t) \) around \( S_{t-1} \),
  we get
  \[ g(S_t) \approx g(S_{t-1}) + \{ J(g(S_t)) \} (S_t - S_{t-1}) \]
where \( a \) \( \Delta \) \( \nabla g(S_t) \) \( \nabla g(S_{t-1}) \) \( \{ J(g(S_t)) \} \) is a Jacobian matrix.

• We adapted from the algorithm in [2, Sec. 3.3], originally developed for the matrix Bingham-Von Mises-Fisher p.d.f.
• Under the restriction that \( B \) is a block-diagonal matrix,
  \[ \prod_{i=1}^{k} \text{MFB}(A_i, m_i | m_i B_i (m_i B_i)^{-1}) \]
• As a result of (2):
  \[ p(S_t | Y_t, S_t^{[0]}) \approx \text{FIB}(S_t | A_t, B_t) \]
• The weights are then exactly propagated as
  \[ w_{t+1} \propto w_t \rho(Y_t | \text{FIB}(S_t | A_t, B_t)) \]

• To update the weights (Equation 3), it is necessary to compute the normalization constants
  \( cFIB(A_t, B_t) \approx \int S_t \rho(Y_t | \text{FIB}(S_t | A_t, B_t)) dS_t \)
As a result of (2):
  \[ cFIB(A_t, B_t) = \int \rho(Y_t | \text{FIB}(S_t | A_t, B_t)) dS_t \]

6. Computation of the weighted averages on the Steifel manifold
• Ideally, one would estimate the state as a Karcher mean, i.e., the value of \( S_t \) that minimizes the weighted mean square geodesic distance to the particle set.
• To reduce computational complexity, we evaluated the weighted averages over the Steifel manifold as
  \[ S_t^{[i]} = \text{argmin} \left\{ \sum_{i=1}^{N} w_{i} \| \text{vec}(S_t - \text{vec}(S_i)) \|_\text{F} \right\} \]

7. Numerical Experiment
• We performed numerical simulations with 158 independent trails of 100 synthetic data samples. Particle filters used 303 particles.

8. Conclusions
• For certain choices of \( G \), the proposed method outperforms that of [1] at the expense of increased computational complexity.
• Most of the computational complexity of the proposed method is related to drawing samples from and computing normalization constants for the matrix Fisher-Bingham density.

References