Optimal Stopping Times for Estimating Bernoulli Parameters with Applications to Active Imaging

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Introduction

Estimating the parameter of a Bernoulli process $p$ is a fundamental statistical problem with many applications, e.g.: Photon-efficient active imaging

- Conventional systems: nonadaptive, i.e. number of trials fixed a priori.
- Alternative system: data-dependent stopping, known as sequential estimation.
- Motivation: understanding whether such adaptive systems improve estimation performance.
- Goal: Given some trial budget constraint, devise an optimal stopping strategy under a mean-squared error loss function.

Contributions

- Propose a stopping rule through a greedy algorithm that seeks for the least achievable error.
- Generalize stopping rule to a rectangular array of Bernoulli processes, representing pixels in a natural scene.
- Demonstrate a 4.45 dB improvement in simulated active imaging scenarios.

A Single Bernoulli Process

Probability of continuing observations after trial $t$:

$$
\pi_t: \{0, 1\}^t \rightarrow [0, 1], \quad t = 0, 1, \ldots
$$

Number of observed trials $T$ satisfies $\mathbb{E}[T] \leq n$.

- Any stopping rule can be represented by a sequence of continuation probabilities.

Adaptive Stopping Rule

Under Beta($\alpha, \beta$) prior, observing $k$ successes in $m$ trials yields a Beta($\alpha + k, \beta + m - k$) distribution.

Bayes risk reduction from one additional trial:

$$
\Delta R(k, m; \alpha, \beta) = \frac{(\alpha + k)(\beta + m - k)}{(\alpha + \beta + m)(\alpha + \beta + m + 1)}
$$

Proposed data-adaptive stopping rule

Stop when Bayes risk reduction $\Delta R(k, m; \alpha, \beta)$, for an additional trial, is below a specified threshold.

- Visualization of a stopping rule $\rightarrow$ Binary tree with continuation probability labels.
- All observation sequences with $k$ successes in $m$ trials yield the same continuation probability $q_{k,m} \rightarrow$ Trellis.

Arrays of Bernoulli Processes

Scene raster-scanned using pulsed illumination guided by proposed stopping rule.

- Data: arrays of number of pulses $[m_{i,j}]_{i,j}$ and detections $[k_{i,j}]_{i,j}$.
- Reconstruction: TV-regularized ML estimation to exploit spatial correlations.

Results

- True image reflectivity in $[0.001, 0.101]$.
- Beta(2, 152) prior assumed. Trial budget $n = 200$.
- Binomial (Fixed) vs. Adaptive (proposed)

<table>
<thead>
<tr>
<th>Method</th>
<th>MSE $\times 10^{-5}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Binomial (Fixed)</td>
<td>3.43e-05</td>
</tr>
<tr>
<td>Adaptive (proposed)</td>
<td>1.26e-05</td>
</tr>
</tbody>
</table>

Conclusion

- Proposed adaptive stopping rule that yields significant improvements over non-adaptive rule.
- Binomial and Negative Binomial stopping strategies are rarely optimal.

References