1. Background, Motivation and Our Contributions

- NOMA has been a key enabling technology to meet the requirements of 5G on high spectral efficiency, massive connectivity, and low transmission latency;
- Most existing NOMA designs assumed Gaussian inputs. The drawbacks are:
  - the implementation in reality will result in huge storage capacity, unaffordable computational complexity and extremely long encoding/decoding delay;
  - the actual transmitted signals in real communication systems are drawn from finite-alphabet constellations, such as PAM, QAM, and PSK;
- Applying the results derived from the Gaussian inputs to the signals with finite-alphabet inputs can lead to significant performance loss.
- We consider the NOMA design for a classical two-user MAC with QAM constellations at both transmitters, whose sizes are not necessarily the same.
- We aim to maximize the minimum Euclidean distance of the received sum-constellation for a ML receiver where the formulated problem is a mixed continuous-discrete optimization problem and is non-trivial to resolve;
- We discover that Farey sequence can be employed to tackle the formulated problem. However, the existing Farey sequence is not applicable when the constellation sizes of the two users are different;
- To address this challenge, we define a new type of Farey sequence, termed punched Farey sequence. Based on the punched Farey sequence and its properties, we manage to resolve the mixed-continuous-discrete optimization problem by providing a neat closed-form optimal solution.

2. System Model and Problem Formulation

![Figure: Two-User Real Gaussian Multiple Access Channel](image)

The received signal at the access point D can be written as:

\[ y = [h_1]s_1 + [h_2]s_2 + n, \]

where \( s_2 \in \{ \pm (2^{k_2}-1) \}_2^{M_2} \) and \( k_2 = 2 \) are drawn from a standard PAM constellation with equal probability. \( 0 < |h_1| \leq 1 \) and \( 0 < |h_2| \leq 1 \) are the weighting coefficients;

A coherent maximum-likelihood (ML) detector is used by the access point D to estimate the transmitted signals in a symbol-by-symbol fashion. Mathematically, the estimated signals can be expressed as:

\[ \hat{s}_1, \hat{s}_2 = \arg \min_{s_1, s_2} |y - (|h_1|s_1 + |h_2|s_2)|. \]

The minimum Euclidean distance between two constellations are given by:

\[ d(m, n) = \left\| [h_1]m + [h_2]n \right\|, \]

where \( m, n \in \mathbb{Z}_2^{(M_1-1, M_2-1)} \) \( \setminus \{0, 0\} \).

3. The Design Problem for the Finite-Alphabet NOMA

**Problem 1:** Find the optimal \((\hat{u}_1^*, \hat{u}_2^*)\) subject to the individual average power constraint such that the minimum Euclidean distance \(d^*\) of the received constellation points is maximized, i.e.,

\[ (\hat{u}_1^*, \hat{u}_2^*) = \arg \max_{(u_1, u_2) \in \mathbb{C}^{2(M_1-1, M_2-1)} \setminus \{0, 0\}} \min_{(m, n)} d(m, n) \]

\[ 0 < |h_1| \leq 1 \text{ and } 0 < |h_2| \leq 1. \]

To that end, we should solve the following optimization problem first:

**Problem 2:** Find the optimal \((\hat{u}_1^*(k), \hat{u}_2^*(k))\) such that

\[ g_{\hat{u}_1^*(k)} \left( b_{\hat{u}_1^*(k)} \right) = \max \min_{(m, n)} d(m, n) \]

where

\[ b_{\hat{u}_1^*(k)} = \left\{ \left( \frac{h_1}{\hat{u}_1^*(k)} - \frac{b_{\hat{u}_1^*(k)}}{h_2} \right), 0 < |\hat{u}_1^*(k)| \leq 1 \right\}. \]

The punched Farey sequence given by \( \mathcal{F}^{M_2-1}_{k_2} = \{ \frac{b_{\hat{u}_1^*(k)} h_1}{h_2}, \ldots, \frac{b_{\hat{u}_1^*(k)} h_1}{h_2} \} \) whose definition will be elaborated in the following part.

4. Punched Farey Sequence

We now propose a new definition in number theory called Punched Farey sequence which characterizes the relationship between two positive integers:

**Definition:** The punched Farey sequence \( \mathcal{F}^{k_2}_{M_2} \) is the ascending sequence of irreducible fractions whose denominators are no greater than \( k_2 \) and numerators are no greater than \( L \).

**Example:** \( \mathcal{F}^{2}_{3} \) is the ordered sequence \( \{0, \frac{1}{3}, \frac{2}{3}, \frac{1}{2}, \frac{2}{5}, \frac{1}{1}\} \).

**Properties:**

- If \( \frac{m_1}{m_2} \) and \( \frac{m_2}{m_3} \) are two adjacent terms in \( \mathcal{F}^{k_2}_{M_2} \) (\( \min \{ K, L \} \geq 2 \)) such that \( \frac{m_1}{m_2} < \frac{m_2}{m_3} \) and \( \frac{m_1}{m_2} \in (\frac{m_2}{m_3}, \frac{m_2}{m_3} + 1) \), then \( m_1 m_3 - m_2^2 = 1 \).
- If \( m_1 + m_3 > K \) and \( m_1 + m_3 \leq K \), then \( m_1 + m_3 - K > L \); if \( n_1 + n_2 > L \), then \( n_1 + n_2 - L > 1 \) where the equality is attained if and only if \( m_1 = m_2 = \frac{k_2}{2} \). Likewise, \( m_1 + m_3 > K \) and \( m_1 + m_3 - K = 1 \) also holds.
- If \( m_1, m_2, m_3, m_4 \) are three consecutive terms in \( \mathcal{F}^{k_2}_{M_2} \) with \( \min \{ K, L \} \geq 2 \) such that \( \frac{m_1}{m_2} < \frac{m_2}{m_3} < \frac{m_3}{m_4} \), then \( m_1 m_4 = m_2 m_3 - 1 \).

**Lemma 1:** The optimal solution to Problem 2 is given as follows:

If \( |h_1| < \frac{b_{\hat{u}_1^*(k)} h_1}{h_2} \), then \( g_{\hat{u}_1^*(k)} \left( b_{\hat{u}_1^*(k)} \right) = \frac{b_{\hat{u}_1^*(k)} h_1}{h_2} \) and \( (u_1^*(k), u_2^*(k)) = (\frac{b_{\hat{u}_1^*(k)} h_1}{h_2}, \frac{b_{\hat{u}_1^*(k)} h_1}{h_2}) \).

5.1 The Solution to Problem 2

We now can solve Problem 2, i.e., restricting \( \frac{b_{\hat{u}_1^*(k)} h_1}{h_2} \) into a certain punched Farey interval determined by the corresponding Farey pair where a closed-form solution is attainable.

5.2 The Solution to Problem 1

**Theorem:** Closed-form optimal weighting coefficients: The optimal solution to Problem 1 in terms of \((u_1^*, u_2^*)\) is given by:

- If \( |h_1| \leq \frac{1}{\sqrt{2M_1-1}} \) and \( |h_2| \leq \frac{1}{\sqrt{2M_2-1}} \), then \( (u_1^*, u_2^*) = \left( \frac{\sqrt{2M_2-1}}{2(M_1-1)}, \frac{\sqrt{2M_2-1}}{2(M_2-1)} \right) \).

6. Numerical Results

![Figure: Comparison of the Proposed-NOMA, CR-NOMA, TDMA and FDMA methods in Ryleigh fading with 8-PSK](image)

Each user adopts 64-QAM for the proposed NOMA design and 64-PSK is used by each user in CR-NOMA. Meanwhile, for TDMA and FDMA methods, each user uses 4096-QAM.

From Fig. (1a) and Fig. (1b), the proposed NOMA design outperforms all the benchmark designs in moderate and high SNR regimes. The FDMA method has a better error performance than the TDMA scheme as expected. The CR-NOMA has the highest BER since the PSK constellation has a smaller Euclidean distance under the same power constraint compared with QAM constellation.

7. Reference