Neighborhood-Preserving Translations on Graphs

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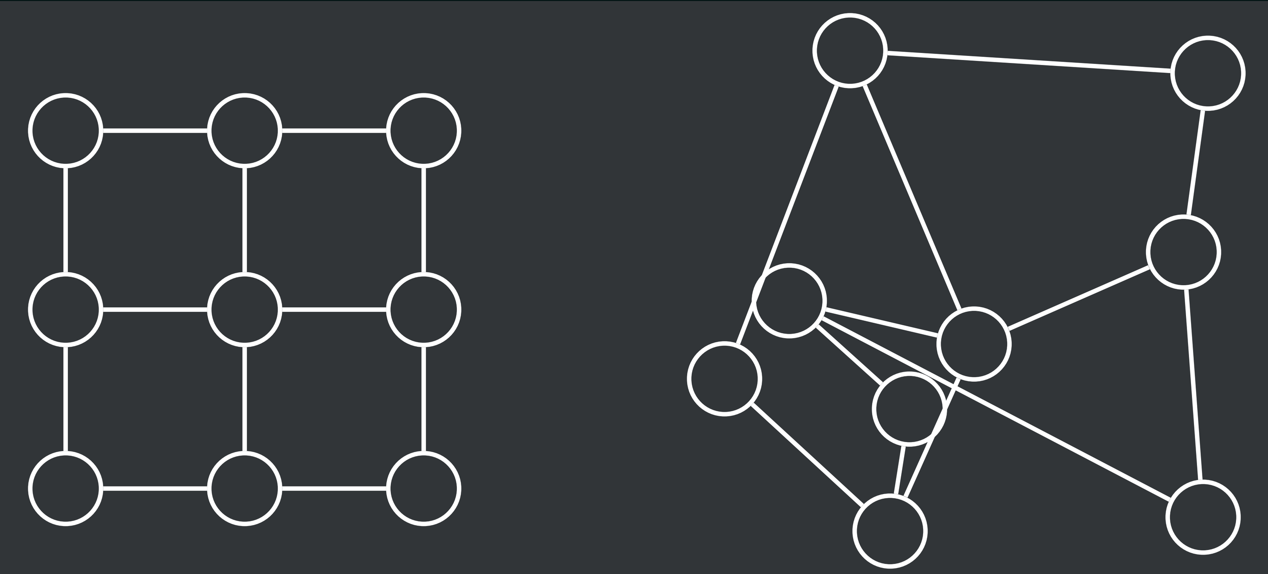
1. Motivation



Define translations in the vertex domain of graphs that matches usual ones on grid graphs.

2. Related Work

- Use Laplacian $L = D - W$, obtain $L = U\Lambda U^\top$, then $U(U^\top \mathbf{x_1} \odot \mathbf{U}^\top \mathbf{x_2})$ is an operator similar to a translation. [4]
- Definition of an isometric operator with respect to the ℓ_2-norm. The proposed translation of a signal consists in multiplying the signal by some matrix which is an exponential of an imaginary diagonal matrix. [5]
- Some definitions of translation exist for powers of a ring graph. [7]

3. Proposed Definitions

Perfect Graphical Translation

An application $f : \mathcal{V} \to \mathcal{V}$ is a perfect graphical translation if:

- f is bijective;
- for all vertex v in $\mathcal{V}$, $f(v)$ is a neighbor of v;
- for all couple $(v_1, v_2) \in \mathcal{V}^2$, $(v_1, v_2) \in \mathcal{E} \Leftrightarrow (f(v_1), f(v_2)) \in \mathcal{E}$.

Candidate Graphical Translation

An application $f : \mathcal{V} \to \mathcal{V} \cup \{\omega\}$ is a candidate graphical translation if:

- $f_{|\mathcal{V}_{0,f}}$ is injective;
- for all vertex v in $\mathcal{V}_{0,f}$, $f(v)$ is a neighbor of v;
- for all couple $(v_1, v_2) \in \mathcal{V}_{0,f}$, $(v_1, v_2) \in \mathcal{E} \Leftrightarrow (f(v_1), f(v_2)) \in \mathcal{E}$.

Generalized Graphical Translation

An application $f : \mathcal{V} \to \mathcal{V} \cup \{\omega\}$ is a generalized graphical translation if:

- f is a candidate graphical translation;
- $\forall v \in \mathcal{V}_{0,f}$, for every g candidate graphical translation such that $g(v) = f(v)$, $|\mathcal{V}_{0,g}| \leq |\mathcal{V}_{0,f}|$.

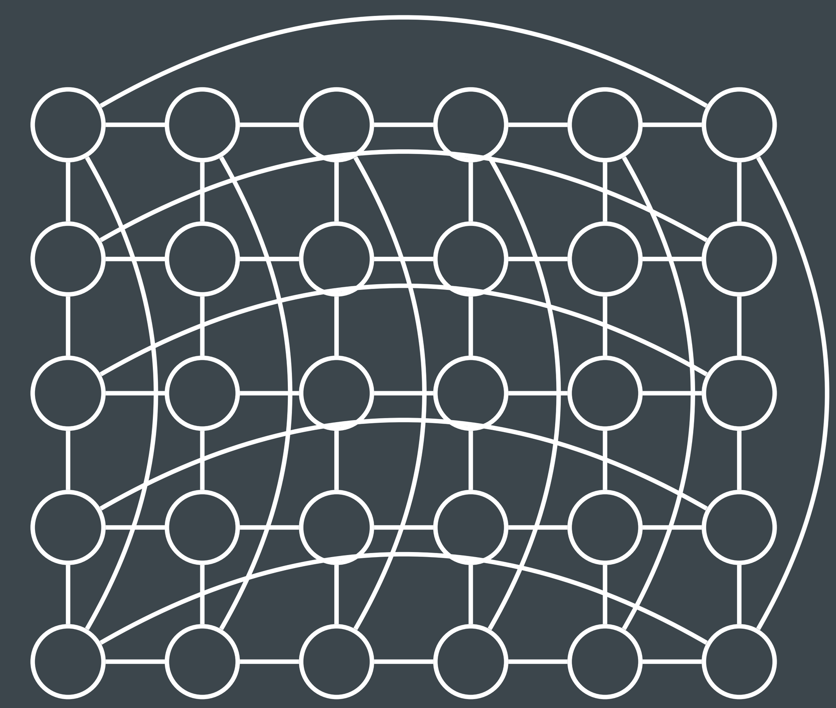
Geometrical Translation

An application $f : \mathcal{V} \to \mathcal{V} \cup \{\omega\}$ is a geometrical translation if there is $\delta = \pm \mathbf{e}_i$ such that

$$\forall \mathbf{v} \in \mathcal{V}, f(\mathbf{v}) = \begin{cases} \mathbf{v} + \delta & \text{if } \mathbf{v} + \delta \in \mathcal{V} \\ \omega & \text{otherwise} \end{cases}.$$

4. Analysis on grid graphs

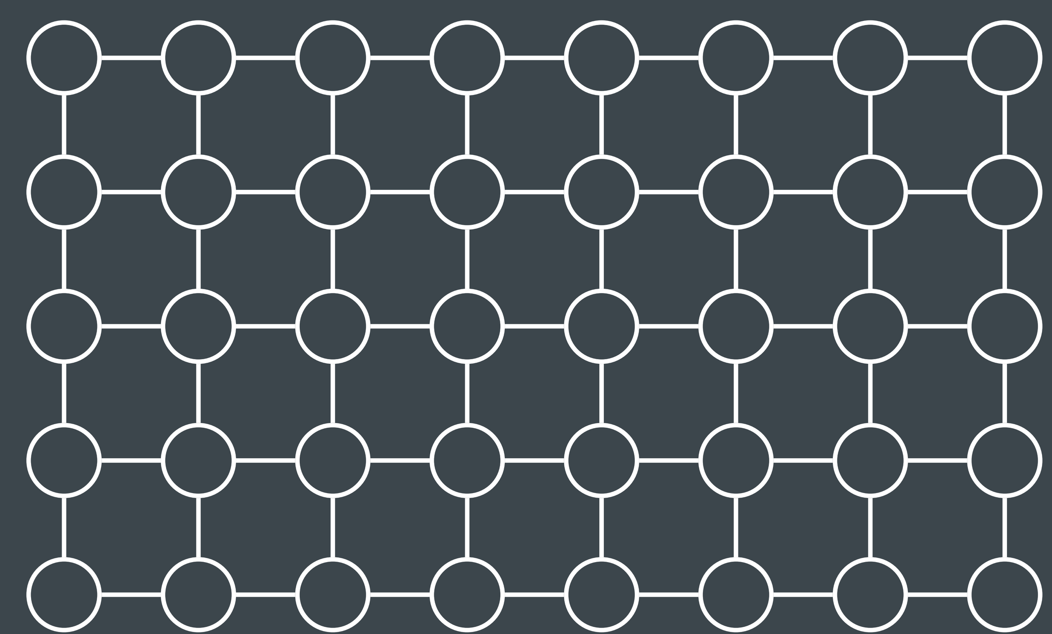
4.1. Analysis on cyclic grid graphs



Proposition: Let f be a perfect graphical translation on a cyclic grid graph, then f is a geometrical translation.

Sketch of proof: Let us denote $\delta = f(\mathbf{0}) - \mathbf{0} = f(\mathbf{0})$ and let $\mathbf{v} \in \mathcal{V}$. As a cyclic grid graph is connected, there is a path from $\mathbf{0}$ to $\mathbf{v}$. By propagating the contamination lemma along the path, neighbor after neighbor, we obtain that $f(\mathbf{v}) = \mathbf{v} + \delta$.

4.2. Analysis on noncyclic grid graphs



Let us consider a noncyclic grid graph with parameters d and $(\ell_i)_{1 \leq i \leq d}$ such that:

$$\forall i \in [\![1, d-1]\!], \ell_i \geq (2 \prod_{k=i+1}^{d} \ell_k) + 2.$$

Moreover, we force $\ell_d \geq 3$.

Remark: the function f_ω is a graphical translation.

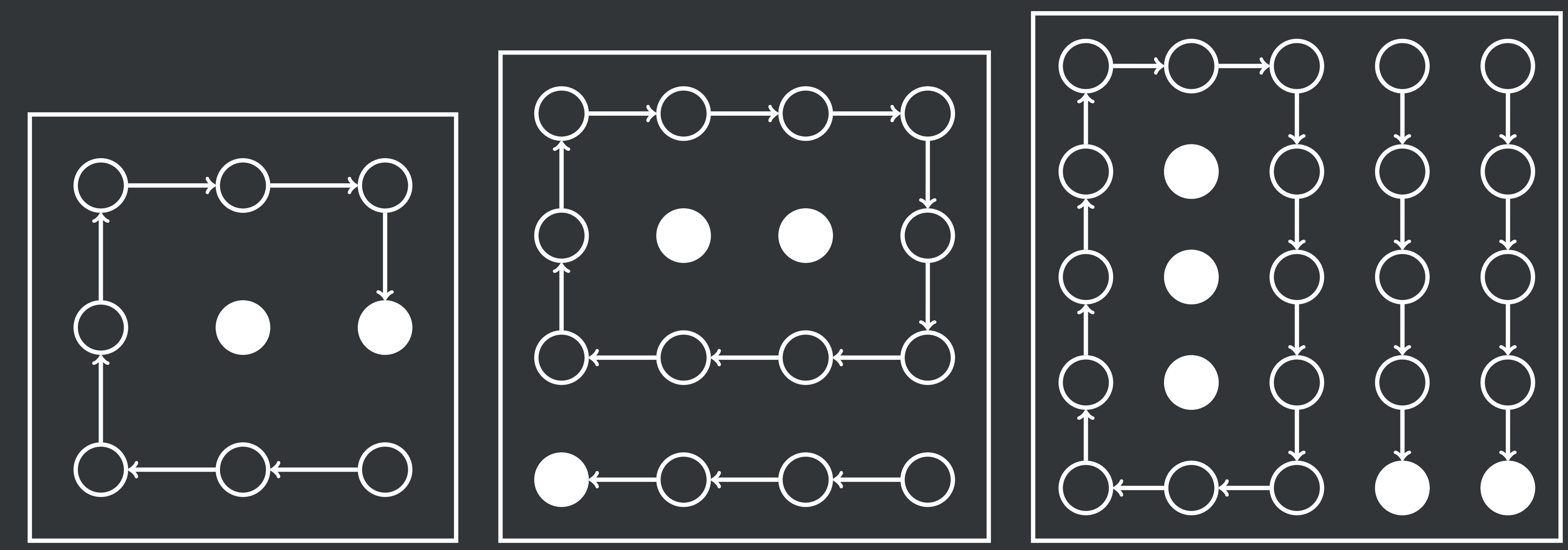
$$f_\omega : \begin{cases} \mathcal{V} \to \mathcal{V} \cup \{\omega\} \\ \mathbf{v} \mapsto \omega \end{cases}$$

A graph may admit several graphical translations f with various sizes of $\mathcal{V}_{0,f}$. We refer to them as c-graphical translations where $c = |\mathcal{V}_{0,f}|$.

Proposition: Graphical translations on such noncyclic grid graphs are geometrical translations and f_ω

5. Counter Examples

On small graphs, one can find applications that verify our definitions without being geometrical translations.



6. Demonstrations

Contamination Lemma

Let f be a perfect graphical translation on a cyclic grid graph $\mathcal{G} = \langle \mathcal{V}, \mathcal{E} \rangle$. Let $\mathbf{v}$ be in $\mathcal{V}$ and consider $\delta = f(\mathbf{v}) - \mathbf{v}$, then $\forall \mathbf{w} \in \mathcal{V}$ neighbor of $\mathbf{v}, f(\mathbf{w}) = \mathbf{w} + \delta$.

Sketch of proof for noncyclic grid grid graphs

- Let f be a candidate graphical translation of any graph $\mathcal{G} = \langle \mathcal{V}, \mathcal{E} \rangle$. For any vertex $\mathbf{v} \in \mathcal{V}$, the sequence $(f^n(\mathbf{v}))_n$ is either periodic or finite (in which case the last element is ω).
  We call grid graph slice $(a, i)^\perp$ a set of vertices that share one coordinate of value a at dimension i.
- Let us consider a noncyclic grid graph $\mathcal{G} = \langle \mathcal{V}, \mathcal{E} \rangle$ with parameters d and $(\ell_i)_{1 \leq i \leq d}$, and f a c-graphical translation with the largest c. Then we have $|\mathcal{V}_{0,f}| \geq (\ell_1 - 1) \prod_{k=2}^{d} \ell_k$.
- Consider a union of two adjacent slices $\mathcal{S}$. Consider f to be a c-graphical translation with the largest c. If there exists some vertex $\mathbf{v} \in \mathcal{V}$ such that $(f^n(\mathbf{v}))_n$ is periodic, then $\mathcal{S} \not\subset \mathcal{V}_{0,f}$.
- Let us consider f to be a c-graphical translation with the largest c. Then there exists m such that $(m, 1)^\perp \subset \mathcal{V}_{0,f}$ and $(m+1, 1)^\perp \subset \mathcal{V}_{0,f}$. Moreover, $f\left((m,1)^\perp \cup (m+1,1)^\perp\right) \not\subset (m,1)^\perp \cup (m+1,1)^\perp$.
- Let us consider f to be a c-graphical translation with the largest c. Then f is the geometrical translation by $\mathbf{e_1}$ or $-\mathbf{e_1}$.

7. Conclusions

- Success for cyclic grid graphs and subfamilies of noncyclic grid graphs.
- Extensions for all graphs could be found by adapting some properties.
- Combinatorial issues to effectively find translations on a given graph.

References

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