DESIGN OF OPTIMAL ENTROPY-CONSTRAINED UNRESTRICTED POLAR QUANTIZER FOR BIVARIATE CIRCULARLY SYMMETRIC SOURCES

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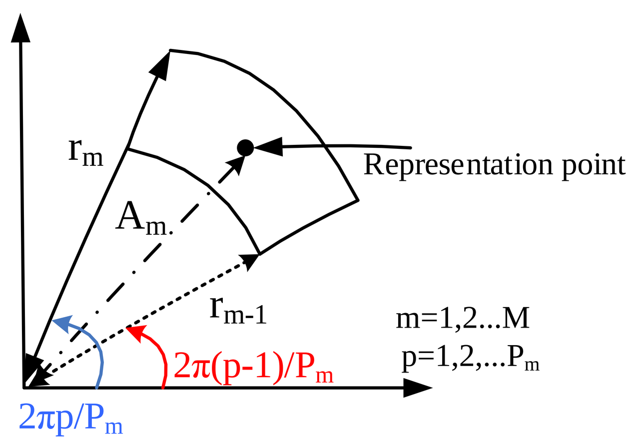
Contribution: Globally Optimal ECUPQ Design

- A globally optimal entropy-constrained unrestricted polar quantizer design algorithm is proposed for finite rates.
- Best practical performance for a bivariate memoryless Gaussian source at small rates known to date.
- The problem of rate allocation between the magnitude and phase quantizers is handled efficiently.
- Journal version is available at [1].

Problem Formulation



A UPQ quantization bin in the polar coordinates

$\mathcal{R}(m,s) = \left\{ re^{j\theta} | r_{m-1} \le r < r_m, (s-1)\frac{2\pi}{P_m} \le \theta < s\frac{2\pi}{P_m} \right\}$.

- M: Number of magnitude levels of the UPQ;
- $\mathbf{r} \triangleq (r_1, r_2, \cdots, r_{M-1})$: The vector of thresholds of the magnitude quantizer;
- P_m: Number of phase regions of the phase quantizer corresponding to cell $[r_{m-1}, r_m)$, and $\mathbf{P} \triangleq (P_1, P_2, \cdots, P_M)$.

The reconstruction for quantizer bin $\mathcal{R}(m,s)$:

$$\theta_{m,s} = (2s-1)\pi/P_m, \quad A_m = sinc\left(\frac{1}{P_m}\right)\frac{\int_{r_{m-1}}^{r_m} rg(r)dr}{\int_{r_{m-1}}^{r_m} g(r)dr}.$$

Distortion (per sample pair):

$$D = \int_0^\infty r^2 g(r)dr - \sum_{m=1}^M A_m^2 \int_{r_{m-1}}^{r_m} g(r)dr.$$

Entropy (bits/sample pair):

$$H = \sum_{m=1}^M \int_{r_{m-1}}^{r_m} g(r)dr\left(-\log_2 \int_{r_{m-1}}^{r_m} g(r)dr + \log_2 P_m\right).$$

Optimization Problem

$$\min_{M,\mathbf{r},\mathbf{P}} \quad D + \lambda H,$$

subject to $r_i \in \mathcal{A}, 1 \le i \le M-1, \ \lambda > 0$.

$\mathcal{A} = \{a_1, a_2, \cdots, a_K\}$ is a finite set from which the magnitude thresholds of the UPQ are selected.

The set of solutions to the above problem, when λ varies over $(0,\infty)$, is the set of UPQs such that the corresponding pair (H, D) is on the lower boundary of the convex hull of the set of all possible pairs (H, D).

Graph Model $G = (V, E, w)$

- For fixed cell $[a_u, a_v)$, the optimal number of phase regions is

$$P^*_{[a_u,a_v)} \triangleq \min \arg\min_P \left(-sinc^2\left(\frac{1}{P}\right)\left(\frac{\int_{a_u}^{a_v} rg(r)dr}{\int_{a_u}^{a_v} g(r)dr}\right)^2 + \lambda \log_2 P\right).$$

- **Using the above relation, the cost becomes the sum of costs of individual magnitude cells.**

Mapping between the magnitude partition and a path in a Weighted Directed Acyclic Graph $G = (V, E, w)$:

- Thresholds set $\mathcal{A} \Leftrightarrow$ Vertex set $V = \{0, 1, 2, \cdots, K+1\}$;
- A magnitude cell $[a_u, a_v) \Leftrightarrow$ A directed edge from u to v in $E = \{(u,v) \in V^2 \mid 0 \le u < v \le K+1\}$;
- Encoder partition $\{[0, r_1), \cdots, [r_{M-1}, \infty)\} \Leftrightarrow$ A path from source node 0 to destination node $K+1$;
- Cost $d([a_u, a_v)) + \lambda H([a_u, a_v))$ for cell $[a_u, a_v) \Leftrightarrow$ Weight of each edge (u, v):

$$w(u,v) \triangleq \int_{a_u}^{a_v} g(r)dr\left(-sinc^2\left(\frac{1}{P^*_{[a_u,a_v)}}\right)\left(\frac{\int_{a_u}^{a_v} rg(r)dr}{\int_{a_u}^{a_v} g(r)dr}\right)^2 + \lambda \log_2 \frac{P^*_{[a_u,a_v)}}{\int_{a_u}^{a_v} g(r)dr}\right).$$

- **Cost of the quantizer $D + \lambda H \Leftrightarrow$ Weight of the path;**
- **The optimization problem is equivalent to finding the Minimum Weight Path in G.**

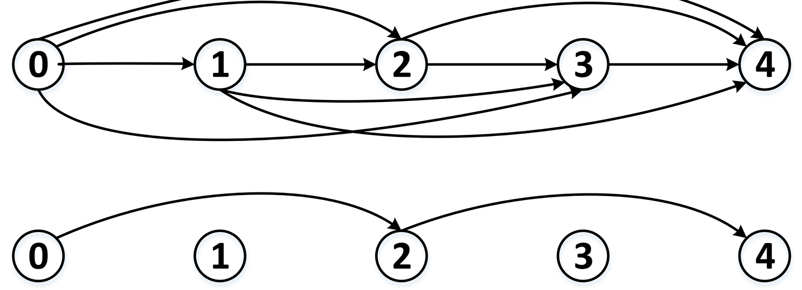


Illustration of the graph G (top) for $K = 3$ and a path in the graph (bottom). Nodes are depicted with circles and edges with arcs. The path shown on the bottom corresponds to the magnitude quantizer with bins $[0, a_2)$ and $[a_2, \infty)$.

Number of Phase Regions: Computation

- The optimal number of phase regions $P^*_{[a_u,a_v)}$ can be found efficiently with the following lemma.
- **Lemma** For each $(u,v) \in E$, the value $P^*_{[a_u,a_v)}$ equals the smallest $P \in \{1, 3, \cdots, P_{max}\}$ [1] satisfying

$$\frac{h(P) - h(P-1)}{g(P) - g(P-1)} \le -\frac{\lambda}{x(a_u,a_v)^2 \ln 2} \le \frac{h(P+1) - h(P)}{g(P+1) - g(P)},$$

where $x(a_u, a_v) = \frac{\int_{a_u}^{a_v} rg(r)dr}{\int_{a_u}^{a_v} g(r)dr}$;

For each pair (a_u, a_v), $0 \le u < v \le K+1$, $P^*_{[a_u,a_v)}$ can be obtained using binary search.

Solution Algorithm: $O(K^2 \log P_{max})$

Proposed Solution Procedure

1. Evaluate cumulative probabilities and first moments during the preprocessing step – $O(K)$ time;
2. For each pair (a_u, a_v), $0 \le u < v \le K+1$, compute the number of phase regions $P^*_{[a_u,a_v)}$ and the edge weight $w(u,v)$ – $O(K^2 \log P_{max})$ operations in all;
3. Solve the MWP problem in G – $O(K^2)$ time.

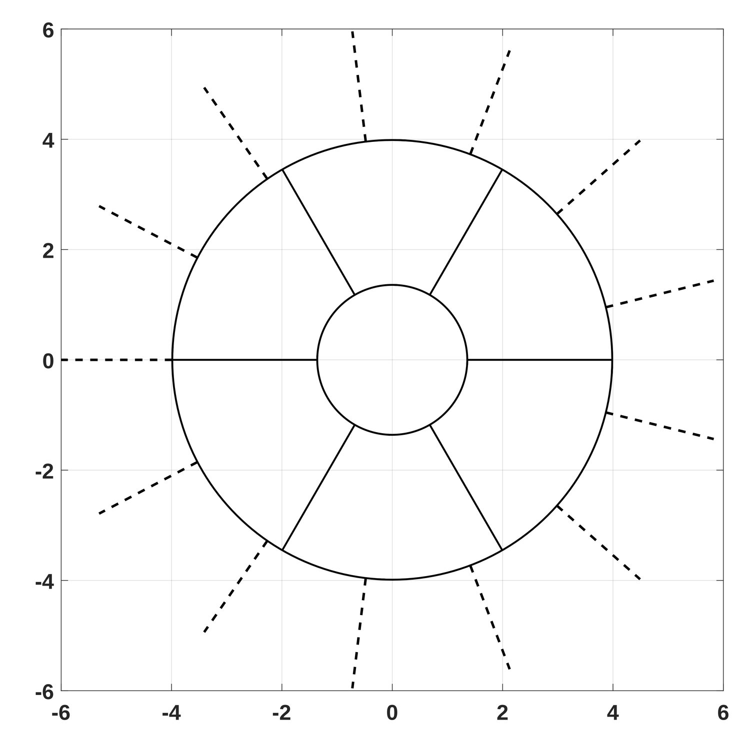
Overall Time Complexity: $O(K^2 \log P_{max})$.

Experimental Results

- 2D Gaussian vector (X_1, X_2), $X_1 \sim \mathcal{N}(0,1)$, $X_2 \sim \mathcal{N}(0,1)$;
- Rate Distortion Function: $D_G(R) = 2 \times 2^{-R}$.

Performance comparison with the entropy-coded UPQ [2] and $D_G(R)$

Rate	Distortion (dB)	Improvement over [2] (dB)	Gap to rate distortion function
1.000	0.883	0.465	0.883
1.585	−0.550	0.216	1.211
2.000	−1.681	0.290	1.328
2.556	−3.295	0.354	1.391
3.139	−4.987	0.716	1.450
3.508	−6.079	0.643	1.473
3.895	−7.224	0.609	1.492
4.512	−9.058	0.755	1.515
4.990	−10.486	0.723	1.527



ECUPQ at $R = 2.0$ bits/sample pair .

Performance comparison with ASY, PASY in [3] and $D_G(R)$
ASY: Asymptotical ECUPQ performance derived in [3]
PASY: Practical ECUPQ based on the asymptotic point density functions

Rate	Distortion (dB)	Distortion of ASY (dB)	Distortion of PASY (dB)	Improvement over PASY (dB)	Gap to rate distortion function
4.100	−7.832	−7.800	−7.213	0.619	1.501
4.512	−9.058	−9.041	−8.481	0.577	1.515
4.990	−10.486	−10.480	−9.973	0.513	1.527
5.996	−13.500	−13.507	−13.144	0.356	1.539
6.995	−16.506	−16.514	−16.287	0.219	1.541
8.000	−19.532	−19.539	−19.408	0.124	1.540
9.000	−22.547	−22.549	−22.481	0.066	1.538
9.990	−25.528	−25.530	−25.493	0.035	1.536
10.992	−28.544	−28.546	−28.528	0.016	1.536
11.991	−31.550	−31.553	−31.542	0.008	1.536

The proposed algorithm

- always outperforms the design of [2], with gains larger than 0.6 dB when $R \ge 3$.
- outperforms PASY for all rates examined, with improvements higher than 0.513 dB when $R \le 4.99$.
- performs extremely close to ASY when $R \ge 4.99$, and is slightly better at lower rates.
- achieves the gap to the rate distortion function very close to the gap (1.529 dB) predicted by the high-resolution quantization theory for the case of single encoder-decoder scalar quantizer.

References

[1] H. Wu and S. Dumitrescu, “Design of optimal entropy-constrained unrestricted polar quantizer for bivariate circularly symmetric sources”, to appear in *IEEE Trans. Commun.*.
[2] S. G. Wilson, “Magnitude/phase quantization of independent Gaussian variates”, *IEEE Trans. Commun.*, vol. COM-28, no. 11, pp. 1924-1929, Nov. 1980.
[3] R. Vafin and W. B. Kleijn, “Entropy-constrained polar quantization and its application to audio coding”, *IEEE Trans. Speech and Audio Process.*, vol. 13, no. 2, pp. 220-232, Mar. 2005.