Definition of Query Procedure

Given any \( n \in \mathbb{N} \) and \( \epsilon \in [0, 1] \), to search for a moving target over a \( d \)-dimensional torus, \((n, \epsilon, d, \delta)\)/non-adaptive query procedure consists of:

- \( n \) queries \( A^n \), where for each \( i \in [n] \), we query whether the target locates inside a Lebesgue measurable subset \( A_i \in \{0, 1\}^d \) and
- a decoder \( Y^* \), where \( \Pr \left( Y^* = 0 \mid n, \epsilon, d, \delta \right) \)

such that the excess-resolution probability satisfies

\[
P_e(n, \epsilon, d, \delta) = \sup_{n, \epsilon, d, \delta} \Pr \left( \max_{i \in [n]} \left( Y^*_i - \mathbb{E}[Y^*_i] \right) > \delta \right) \leq \epsilon.
\]

Accurate estimation of the trajectory implies accurate estimate of the initial location and velocity, and vice versa

- \( |S_i - S_i^0| < \frac{\delta}{n} \) and \( |V_i - V_i^0| < \frac{\delta}{2} \) imply accurate estimate of the trajectory, i.e., \( \text{meas}_{\epsilon, \delta, S_0} \left( S_i \right) \approx S_i \approx V_i \approx V_i \approx \delta \)

- \( |S_i - S_i^0| > \frac{\delta}{n} \) or \( |V_i - V_i^0| > \frac{\delta}{2} \) implies poor estimate

Fundamental Limit

- Given any number of queries \( n \in \mathbb{N} \) and \( \epsilon \in [0, 1] \),

\[
\delta^n(n, \epsilon, d, \delta) \equiv \inf \left\{ \epsilon \in [0, 1] \cap \mathbb{Q} \mid Q^n(n, \epsilon, d, \delta) \right\}
\]

- minimal non-asymptotic resolution achievable by any non-adaptive query procedure with \( n \) queries and excess-resolution probability \( \epsilon \)

Dual quantity (sample complexity):

\[
n^* \equiv n^*(d, \delta, \epsilon) \equiv \inf \left\{ n \in \mathbb{N} \mid \delta^n(n, \epsilon, d, \delta) \leq \delta \right\}
\]

Preliminaries

- \( P_X = \text{Bernoulli}() \) denotes the Bernoulli distribution

\[
\text{Bernoulli}(p) \sim \text{Bernoulli}(1) \quad \text{when } |X| = |\epsilon|
\]

- \( P_{Y^n|S} \) denotes the distribution on \( Y^n \) induced by \( P_X \) and \( P_{Y|S} \)

- For any \( (x, y) \in X \times Y \), define the mutual information density

\[
I_p(x, y) = \log p(Y|x) - \log p(Y)
\]

- “Capacity” of measurement dependent channels \( P_{Y^n|S} \)

\[
C = \max_{p_{\epsilon|s}} I_p(X, Y) \quad \text{w.r.t. } p_{\epsilon|s}
\]

- “Dispersion” of measurement dependent channels

\[
V = \min_{p_{\epsilon|s}} I_p(X, Y) \quad \text{w.r.t. } p_{\epsilon|s}
\]

Main Result

Theorem 1 For any \( \epsilon \in (0, 1) \) and finite \( d \in \mathbb{N} \), the minimal achievable resolution \( \delta^n(n, \epsilon, d, \delta) \) satisfies the following properties

- \( n(\epsilon, d, \delta) = O\left(\frac{\log(1+\delta)}{\epsilon}\right) \)

- \( \text{Measure}^{\delta^n(n, \epsilon, d, \delta)} + O(n(\epsilon, d, \delta)) \)

- \( n(\epsilon, d, \delta) = O\left(\frac{\log(1+\delta)}{\epsilon}\right) \)

Discussions

- Theorem 1 is tight under maximal speed constraint \( v_{\max} \)

- Refines the result by Kaspi et al. TIT 2018 (Theorem 3):

- Non-asymptotic, non-vanishing vs asymptotic, vanishing

- Any measurement dependent channel vs a measurement dependent BSC

- Multidimensional vs one-dimensional

- Strong converse holds

\[
\lim_{n \to \infty} \frac{1}{n} \log P(X^n, Y^n) \leq C
\]

- Proof ideas: finite blocklength channel coding + analysis of the number of quantized trajectories

References


