Estimating Correlation Coefficients for Quantum Radar and Noise Radar
A Simulation Study

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What is a quantum radar?

- Any radar that exploits phenomena from quantum physics to improve detection performance
- But quantum physics includes everything in classical physics!
- We look at a distinctively quantum phenomenon called entanglement
What entanglement is not

Some people explain entanglement like this:

“If one entangled particle interacts with something, its twin would react in the same way, even if it is far away.”
What entanglement is **not**

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**NO.**
What entanglement *is*

All you need to understand for quantum radar:

entanglement = strong correlation
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Correlation $\implies$ probability theory. Does entanglement involve probability, statistics, random variables?
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Correlation $\implies$ probability theory. Does entanglement involve probability, statistics, random variables?

YES!
Quantum radar: the basic idea

1. Produce a pair of entangled microwave beams.
2. Transmit one of the beams. Keep the other.
3. Receive and measure the signal.
4. Correlate the received and retained signals. Declare a detection if the correlation exceeds a certain threshold.
Quantum radar: the basic idea

- Entangled signals are highly correlated at transmitter

- **High** correlation at receiver $\implies$ target present

- **Low** correlation at receiver $\implies$ target absent
But you don’t need quantum for this, right?

- Isn’t this just **matched filtering**?
- Can’t we generate 100% correlated signals?
- Why bother with all this quantum stuff?
The bad news: quantum noise

- Conventional matched filtering assumes a perfect copy of the signal is available.

- Quantum mechanics says a perfect copy is impossible.

- There will always be noise in $I$ and $Q$ voltage measurements, even in a theoretically ideal system.

- Quantum noise exists even at absolute zero and in a perfect vacuum.
Can’t you just split the signal?
Can’t you just split the signal?

- **Vacuum noise** will creep into the beamsplitter, even at absolute zero and in a perfect vacuum
Quantum noise

- Classically ideal signal: $I(t) = A \cos(\omega t)$
- Quantum ideal signal: $I(t) \sim A \cos(\omega t) + \mathcal{N}(0, \sigma^2)$
Quantum noise

Classically ideal signal: \( I(t) = A \cos(\omega t) \)

Quantum ideal signal: \( I(t) \sim A \cos(\omega t) + \mathcal{N}(0, \sigma^2) \)

- Gaussian noise with power depending only on \( \omega \)
Quantum noise and entanglement

100% correlation is **impossible** between signals with uncorrelated quantum noise
- No such thing as perfect matched filtering

Quantum noise cannot be eliminated, but **can be correlated** between two signals
- Better “matched filtering”

![Diagram showing achievable classical correlation and entanglement](image-url)
Quantum two-mode squeezing radar

1. Produce a pair of entangled microwave beams.
2. Transmit one of the beams. Immediately record a time series of I/Q voltages for the other beam.
3. Receive and record I/Q voltages.
4. Perform matched filtering as usual.

Note: a prototype QTMS radar has been built!
The QTMS radar covariance matrix

\[
\begin{bmatrix}
\sigma_1^2 & 0 & \rho \sigma_1 \sigma_2 \cos \phi & \rho \sigma_1 \sigma_2 \sin \phi \\
0 & \sigma_2^2 & \rho \sigma_1 \sigma_2 \sin \phi & -\rho \sigma_1 \sigma_2 \cos \phi \\
\rho \sigma_1 \sigma_2 \cos \phi & \rho \sigma_1 \sigma_2 \sin \phi & \sigma_2^2 & 0 \\
\rho \sigma_1 \sigma_2 \sin \phi & -\rho \sigma_1 \sigma_2 \cos \phi & 0 & \sigma_2^2
\end{bmatrix}
\]

- $I_1$, $Q_1$, $I_2$, $Q_2$ are **Gaussian random variables** characterized by this covariance matrix.

- $\sigma_1^2$, $\sigma_2^2$ are signal powers for the received and recorded signals; $\phi$ is the phase between them.
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\end{bmatrix}
\]

- \( I_1, Q_1, I_2, Q_2 \) are \textbf{Gaussian random variables} characterized by this covariance matrix.

- \( \sigma_1^2, \sigma_2^2 \) are signal powers for the received and recorded signals; \( \phi \) is the phase between them.

- \( \rho \) characterizes the \textbf{correlation} between the two signals.
\( \rho \) as a detector function

- \( \rho > 0 \) at receiver \( \implies \) target present
- \( \rho = 0 \) at receiver \( \implies \) target absent
- Note: entanglement improves \( \rho \) at transmitter
  - Only need to distinguish between \( \rho > 0 \) and \( \rho = 0 \) at receiver
Estimation of $\rho$

\[
\begin{bmatrix}
\sigma_1^2 & 0 & \rho \sigma_1 \sigma_2 \cos \phi & \rho \sigma_1 \sigma_2 \sin \phi \\
0 & \sigma_2^2 & \rho \sigma_1 \sigma_2 \sin \phi & -\rho \sigma_1 \sigma_2 \cos \phi \\
\rho \sigma_1 \sigma_2 \cos \phi & \rho \sigma_1 \sigma_2 \sin \phi & \sigma_2^2 & 0 \\
\rho \sigma_1 \sigma_2 \sin \phi & -\rho \sigma_1 \sigma_2 \cos \phi & 0 & \sigma_2^2
\end{bmatrix}
\]

- Can estimate the covariance matrix from measurement data using the sample covariance matrix

\[
\hat{S} = \frac{1}{N} \sum_{n=1}^{N} x_n x_n^T
\]

- Problem: no guarantee that $\hat{S}$ is of the above form
Estimation of $\rho$

- One way to estimate $\rho$ from the sample covariance matrix:

\[
\min_{\sigma_1, \sigma_2, \rho, \phi} \left\| R_{QTMS}(\sigma_1, \sigma_2, \rho, \phi) - \hat{S} \right\|_F
\]

- $R_{QTMS}(\sigma_1, \sigma_2, \rho, \phi)$ is the theoretical covariance matrix

- This gives us an **estimate** $\hat{\rho}$ of the underlying, "true" correlation $\rho$
Probability distribution of $\hat{\rho}$

- We have found through simulations that the distribution of $\hat{\rho}$ can be **approximated by the Rice distribution**

$$f(x|\alpha, \beta) = \frac{x}{\beta^2} \exp\left(-\frac{x^2 + \alpha^2}{2\beta^2}\right) I_0\left(\frac{x\alpha}{\beta^2}\right)$$

- In terms of the underlying, “true” $\rho$ and the number of integrated samples $N$:

$$\alpha = \rho$$

$$\beta = \frac{1 - \rho^2}{\sqrt{2N}}.$$
Simulated data

ρ = 0.01, N = 10,000

ρ = 0.5, N = 10,000

▶ Orange bars: histograms of \( \hat{\rho} \) obtained from simulations of QTMS radar measurements

▶ Solid curves: Rice distribution approximation
Simulated data

- Orange bars: histograms of $\hat{\rho}$ obtained from simulations of QTMS radar measurements
- Solid curves: Rice distribution approximation
Simulated data

- Orange bars: histograms of $\hat{\rho}$ obtained from simulations of QTMS radar measurements
- Solid curves: Rice distribution approximation
Based on our Rice distribution approximation, we can obtain an explicit **ROC curve formula** for the detection performance of a QTMS radar:

\[
\rho_D(\rho_{FA}|\rho, N) = Q_1 \left( \frac{\rho \sqrt{2N}}{1 - \rho^2}, \frac{\sqrt{-2 \ln \rho_{FA}}}{1 - \rho^2} \right)
\]
ROC curve plots

- $\rho = 0.005, 0.01, 0.015, 0.02, 0.025, 0.03$
- $N = 50,000$

- $\rho = 0.01$
- $N = 25k, 50k, 75k, 100k, 125k, 150k, 175k, 200k$
Conclusion

- Quantum two-mode squeezing (QTMS) radars involve **correlating two signals**

- Can extract a single **correlation coefficient** $\rho$ that depends on whether target is present/absent

- The **Rice distribution** is a good approximation to the distribution of $\hat{\rho}$

- This is a big step toward performance prediction for QTMS radars: we need only focus on determining $\rho$