Class-specific Poisson Image Denoising using Importance Sampling

Milad Niknejad, José M. Bioucas-Dias, Mário A. T. Figueiredo

Instituto de Telecomunicações, Instituto Superior Técnico, University of Lisbon, Portugal
Overview

1. Class-specific image denoising
2. Patch estimation using Monte-Carlo
3. Importance Sampling
4. Applying importance sampling for image denoising
5. Proposed method

Niknejad, Bioucas-Dias, Figueiredo (IST)
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*Examples*: text/document, face, fingerprint, a specific type of medical image (e.g., brain MRI), ...

This knowledge *should be exploited* by the denoising method!
Assumption: A dataset of clean images of the same class is available.
Gaussian noise observation model:

\[ y_i = x_i + v_i \]

\( x_i \) is a patch of the original image; \( y_i \) is the corresponding noisy patch; \( v_i \) is i.i.d. Gaussian noise.
Patch-based image denoising

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Poisson noise observation model (the focus of this presentation):

\[ y_{i,j} \sim \mathcal{P}(x_{i,j}). \]

\( x_{i,j} \) is the \( j^{th} \) pixel of \( x_i \).
\( \mathcal{P} \) is a Poisson distribution with mean \( x_{i,j} \).
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**Goal**: recover the clean patch \( x_i \) from the noisy one \( y_i \).
MMSE (minimum mean squared error) estimation

MMSE patch estimate ($p(y = y_i)$ is replaced by $p(y_i)$):

$$\hat{x}_i = \mathbb{E}[x|y_i] = \int_{\mathbb{R}^p} x \ p(x|y_i) \ dx = \int_{\mathbb{R}^p} x \ \frac{p(y_i|x) \ p(x)}{p(y_i)} \ dx$$

This multi-dimensional integral is intractable, in general (exception: Gaussian noise and Gaussian prior).

Monte-Carlo approximation: obtain samples $x_j$ from $p(x|y_i)$

$$\hat{\hat{x}}_i = \frac{1}{n} \sum_{j=1}^{n} x_j \lim_{n \to \infty} \hat{\hat{x}}_i = \hat{x}_i$$

However, sampling from $p(x|y_i)$ is also intractable.

Can we approximate $\hat{x}_i$ by sampling from another distribution?
MMSE \((\text{minimum mean squared error})\) estimation

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- However, sampling from $p(x|y_i)$ is also **intractable**.
- Can we approximate $\hat{x}_i$ by sampling from another distribution?
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As in plain Monte-Carlo: \( \lim_{n \to \infty} \hat{E}_n[f(z)] = \mathbb{E}[f(z)] \).
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- Let \( \tilde{q}(z) = b \, q(z) \) be another un-normalized density; assume it is possible/easy to obtain samples \( z_1, ..., z_n \sim q(z) \).
- Constants \( c \) and \( b \) may be unknown.

\[ \hat{\mathbb{E}}_n[f(z)] = \frac{1}{n} \sum_{j=1}^{n} f(z_j) \frac{\tilde{p}(z_j)}{\tilde{q}(z_j)}. \]
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$$\hat{\mathbb{E}}_n[f(\mathbf{z})] = \frac{1}{n} \sum_{j=1}^{n} f(\mathbf{z}_j) w(\mathbf{z}_j), \quad w(\mathbf{z}_j) = \frac{\tilde{p}(\mathbf{z}_j)}{\tilde{q}(\mathbf{z}_j)}.$$
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Applying SNIS for MMSE patch estimation

Back to our problem:

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- Instead of sampling from \( p(x|y_i) \), use samples \( x_1, ..., x_n \) from \( p(x) \);

\[ \text{Why?} \quad \tilde{p}(z) = p(y_i|x) p(x), c = 1/p(y_i) \text{ and } \tilde{q}(z) = p(x). \]
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- Instead of sampling from \( p(x|y_i) \), use samples \( x_1, ..., x_n \) from \( p(x) \);
- Simply use samples from the external dataset of clean patches.
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$$\hat{x}_i = \hat{\mathbb{E}}_n[x|y_i] = \frac{\sum_{j=1}^{n} x_j \ w_j}{\sum_{j=1}^{n} w_j}, \quad w_j = p(y_i|x = x_j)$$
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- Why? \( \tilde{p}(z) = p(y_i | x) \ p(x), \ c = 1/p(y_i) \) and \( \tilde{q}(z) = p(x) \).
For Poisson noise, the weights are easy to obtain $(y_{i,j} \sim \mathcal{P}(x_{i,j})$, i.i.d.):

$$wj = \prod_{l=1}^{N} \frac{e^{-x(j,l)}(x(j,l))^{y(j,l)}}{y(j,l)!}$$

It can be adapted to other image restoration tasks, such as deblurring,
Applying SNIS for MMSE patch estimation (II)

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- It can be generalized to other noise models.
Key observations:

1. Using samples from $p(x)$ is sub-optimal, as it may have high variance (or even infinite variance). It requires very large $n$. 

Proposed approach:

1. Cluster the patches in the external dataset.
2. Assign each noisy patch to the closest cluster.
3. Use the corresponding clean patches as samples from the proposal distribution for SNIS.
Key observations:

1. Using samples from $p(x)$ is sub-optimal, as it may have high variance (or even infinite variance). It requires very large $n$.

2. The proposal distribution should be made similar to each target distribution $p(x|y_i)$: Estimator with lower MMSE for limited number of samples.
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Proposed method

- Clustering: Any clustering algorithm can be used (k-means,...). The whole dataset of patches is clustered to $K$ clusters. $\{X_1 \ldots X_K\}$.

Objective is to solve the following simultaneous classification and estimation problem:

$$\hat{(x_i, \hat{k}_i)} = \arg\min_{(u, k)} \int_{\mathbb{R}^m} \|u - x\|^2_2 p(x|y_i, k) \, dx$$

The above chooses the best cluster $\hat{k}_i$, and use this distribution to approximate the integral. It is equivalent to sampling from (unknown) $\hat{k}_i$th distribution as the proposal distribution. The above integral is intractable, but we can use SNIS.
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- It is equivalent to sampling from (unknown) $\hat{k}_i^{th}$ distribution as the proposal distribution.

- The above integral is intractable, but we can use SNIS.
\[ \mathbb{E}[\|x - u\|_2^2 | y_i, k] = \int_{\mathbb{R}_+^m} \|u - x\|_2^2 p(x | y_i, k) \, dx. \]
Proposed method

\[ \mathbb{E}[\|x - u\|^2_2|y_i, k] = \int_{\mathbb{R}^m} \|u - x\|^2_2 p(x|y_i, k) \, dx. \]

Using SNIS, the above can be approximated by

\[ \hat{\mathbb{E}}_n[\|x - u\|^2_2|y_i, k] = \sum_{j_k=1}^{n} \frac{\|u - x_{jk}\|^2_2 w_{jk}}{\sum_{j=1}^{n} w_{jk}} \]

(1)

where the \(x_{jk}\), for \(j_k = 1, \ldots, n\) are samples from the distribution \(p(x|k)\).

\[ w_{jk} = p(y_i|x_{jk}) \]
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We Minimize by alternating minimization

- when \( u = \hat{x}_i \) is fixed,

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\[\hat{x}_i = \mathbb{E}_{n_1}[x | y_i, \hat{k}] = \frac{\sum_{j=1}^{n_1} w_{j\hat{k}} x_{j\hat{k}}}{\sum_{j=1}^{n_1} w_{j\hat{k}}}.\]
Implementation Details

Speeding up the algorithm:

The key to speeding up is to limit the numbers of patch samples $n_1$ and $n_2$.

Clustering: $n_2 = 30$, overall 600 patches for all $k = 20$ clusters (less than 1 percent of samples in external datasets).

Denoising: samples derived for each patch $n_1$ was set to 300.

Overall: 900 patches are processed for each denoised patch (computational complexity is similar to an internal non-local denoising with the patches constrained in $30 \times 30$ window).
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Experiment 1

Noisy image (Peak=10)  Non-local PCA (PSNR=22.60)  VST+BM3D (PSNR=24.79)  Poisson NL means (PSNR=24.55)  Proposed (PSNR=26.40)
Experiment 2

Noisy (Peak=2)
Non-local PCA (PSNR=14.95)
VST+BM3D (PSNR=14.55)
Proposed (PSNR=18.64)
We proposed a method based on importance sampling in which no parametric distribution is fitted to data.
Conclusion

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- The method can be generalized easily to other image restoration inverse problems.
A. Owen
Monte Carlo Theory, Methods and Examples
Available at http://statweb.stanford.edu/ owen/mc/.

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