A Recursive Least-Squares Algorithm Based on the Nearest Kronecker Product Decomposition

Camelia Elisei-Iliescu¹, Constantin Paleologu¹, Jacob Benesty², and Silviu Ciochină¹

¹Department of Telecommunications
University Politehnica of Bucharest, Romania
{pale, silviu}@comm.pub.ro

²INRS-EMT
University of Quebec, Montreal, Canada
benesty@emt.inrs.ca
Outline

• Introduction

• System Model

• RLS algorithm based on the nearest Kronecker product decomposition (RLS-NKP)

• Simulation Results

• Conclusions and Perspectives
Introduction

- **Recursive least-square (RLS)** algorithm \(\rightarrow\) frequently used in system identification problems
  \(\rightarrow\) this algorithm is computationally very complex

- **In this work** \(\rightarrow\) **new approach to improve the efficiency of the RLS** \(\rightarrow\) the impulse response decomposition based on the nearest Kronecker product
  \(\rightarrow\) low-rank approximation

- **Target:** a **high-dimension** system identification problem

  \(\rightarrow\) RLS algorithm based on the nearest Kronecker product decomposition
System Model

Model

\[ d(t) = (\mathbf{h}^T \mathbf{x}(t) + w(t) \]

where \( d(t) \) - desired signal
\( w(t) \) - additive noise

\[ \rightarrow \mathbf{h} \] is the impulse response of the unknown system of length \( L = L_1 L_2 (L_1 \geq L_2) \).

\[ \rightarrow \text{The impulse response can be decomposed as:} \]
\[ \mathbf{h} = [\mathbf{s}_1^T \mathbf{s}_2^T \ldots \mathbf{s}_{L_2}^T]^T \]

where \( \mathbf{s}_l, l = 1, 2, \ldots, L_2 \) - short impulse responses (of length \( L_1 \))

\[ \rightarrow \mathbf{h} \text{ can be approximated as:} \]
\[ \mathbf{h}_2 \otimes \mathbf{h}_1 \]

(h2-length \( L_2 \), h1-length \( L_1 \))

\[ \rightarrow \text{The normalized misalignment:} \]
\[ \mathcal{M} (\mathbf{h}_1, \mathbf{h}_2) = \frac{\| \mathbf{h} - \mathbf{h}_2 \otimes \mathbf{h}_1 \|_2}{\| \mathbf{h} \|_2} \]
System Model

→ We can reorganize the components of \( \mathbf{h} \) into a matrix:
\[
\mathbf{H} = \begin{bmatrix} \mathbf{s}_1 & \mathbf{s}_2 & \ldots & \mathbf{s}_{L_2} \end{bmatrix}
\]

\[
\mathcal{M}(\mathbf{h}_1, \mathbf{h}_2) = \frac{\|\mathbf{H} - \mathbf{h}_1 \mathbf{h}_2^T\|_F}{\|\mathbf{H}\|_F}
\]

→ The optimal values of \( \mathbf{h}_1 \) and \( \mathbf{h}_2 \) minimize \( \mathcal{M}(\mathbf{h}_1, \mathbf{h}_2) \)

→ Minimizing \( \mathcal{M}(\mathbf{h}_1, \mathbf{h}_2) \) is equivalent to finding the nearest rank-1 matrix to \( \mathbf{H} \)

\[
\mathbf{H} = \mathbf{U}_1 \Sigma \mathbf{U}_2^T = \sum_{l=1}^{L_2} \sigma_l \mathbf{u}_{1,l} \mathbf{u}_{2,l}^T
\]
\[
\mathbf{h} \approx \sum_{p=1}^{P} \mathbf{h}_{2,p} \otimes \mathbf{h}_{1,p}
\]
\[
\overline{\mathbf{h}}(P) \approx \sum_{p=1}^{P} \mathbf{h}_{2,p} \otimes \mathbf{h}_{1,p} = \sum_{p=1}^{P} \sigma_p \mathbf{u}_{2,p} \otimes \mathbf{u}_{1,p}
\]

where \( P \leq L_2 \)
System Model

\[ \xi_{12}(h) = \frac{L}{L - \sqrt{L}} \left( 1 - \frac{\|h\|_1}{\sqrt{L} \|h\|_2} \right) \]

(a) Impulse responses with \( L = 500 \) \((L_1 = 25, L_2 = 20)\):
- (a) \( \xi_{12} = 0.8835 \) and \( \text{rank}(H) = 1 \);
- (b) \( \xi_{12} = 0.8835 \) and \( \text{rank}(H) = 4 \).
RLS algorithm based on the nearest Kronecker product decomposition

→ The goal is to estimate $\mathbf{h}$ with an adaptive filter $\hat{\mathbf{h}}(t)$
→ The error signal:

$$e(t) = d(t) - \hat{y}(t) = d(t) - \hat{\mathbf{h}}^T(t-1)\mathbf{x}(t)$$

→ We can decompose the adaptive filter:

$$\hat{\mathbf{h}}(t) = \sum_{p=1}^{P} \hat{\mathbf{h}}_{2,p}(t) \otimes \hat{\mathbf{h}}_{1,p}(t)$$

$$e(t) = d(t) - \sum_{p=1}^{P} \hat{\mathbf{h}}_{1,p}^T(t-1)\mathbf{x}_{2,p}(t) = d(t) - \hat{\mathbf{h}}_{1}^T(t-1)\mathbf{x}_{2}(t)$$

$$e(t) = d(t) - \sum_{p=1}^{P} \hat{\mathbf{h}}_{2,p}^T(t-1)\mathbf{x}_{1,p}(t) = d(t) - \hat{\mathbf{h}}_{2}^T(t-1)\mathbf{x}_{1}(t)$$

$$\mathbf{x}_{2,p} = [\hat{\mathbf{h}}_{2,p}(t-1) \otimes \mathbf{I}_{L_1}]^T \mathbf{x}(t)$$

$$\hat{\mathbf{h}}_{1}(t) = [\hat{\mathbf{h}}_{1,1}(t) \quad \hat{\mathbf{h}}_{1,2}(t) \quad ... \quad \hat{\mathbf{h}}_{1,P}(t)]^T$$

$$\hat{\mathbf{h}}_{2}(t) = [\hat{\mathbf{h}}_{2,1}(t) \quad \hat{\mathbf{h}}_{2,2}(t) \quad ... \quad \hat{\mathbf{h}}_{2,P}(t)]^T$$

$$\mathbf{x}_{1,p} = [\mathbf{I}_{L_2} \otimes \hat{\mathbf{h}}_{1,p}(t-1)]^T \mathbf{x}(t)$$

$$\mathbf{x}_{1}(t) = [\mathbf{x}_{1,1}(t) \quad \mathbf{x}_{1,2}(t) \quad ... \quad \mathbf{x}_{1,P}(t)]^T$$

$$\mathbf{x}_{2}(t) = [\mathbf{x}_{2,1}(t) \quad \mathbf{x}_{2,2}(t) \quad ... \quad \mathbf{x}_{2,P}(t)]^T$$
RLS algorithm based on the nearest Kronecker product decomposition

→ The cost functions:

\[
J_{\hat{h}_2}[\hat{h}_1(t)] = \sum_{i=1}^{t} \lambda_1^{t-i} [d(i) - \hat{h}_1^T(t)x_2(i)]^2
\]

\[
J_{\hat{h}_1}[\hat{h}_2(t)] = \sum_{i=1}^{t} \lambda_2^{t-i} [d(i) - \hat{h}_2^T(t)x_1(i)]^2
\]

\(\lambda_1, \lambda_2\) - forgetting factors

Normal equations

\[
\begin{align*}
R_2(t)\hat{h}_1(t) & = p_2(t) \\
R_1(t)\hat{h}_2(t) & = p_1(t)
\end{align*}
\]

where

\[
R_2(t) = \lambda_1 R_2(t-1) + x_2(t)x_2^T(t)
\]

\[
R_1(t) = \lambda_2 R_1(t-1) + x_1(t)x_1^T(t)
\]

\[
p_2(t) = \lambda_1 p_2(t-1) + x_2(t)d(t)
\]

\[
p_1(t) = \lambda_2 p_1(t-1) + x_1(t)d(t)
\]

→ The RLS-NKP:

\[
\begin{align*}
\hat{h}_1(t) & = \hat{h}_1(t-1) + k_2(t)e(t) \\
\hat{h}_2(t) & = \hat{h}_2(t-1) + k_1(t)e(t)
\end{align*}
\]
RLS algorithm based on the nearest Kronecker product decomposition

→ The Kalman gain vectors:

\[ k_2(t) = \frac{R_2^{-1}(t - 1)x_2(t)}{\lambda_1 + x_2^T(t)R_2^{-1}(t - 1)x_2(t)} \]

\[ k_1(t) = \frac{R_1^{-1}(t - 1)x_1(t)}{\lambda_2 + x_1^T(t)R_1^{-1}(t - 1)x_1(t)} \]

→ The updates of \( R_1^{-1}(t) \) and \( R_2^{-1}(t) \) (based on the inversion lemma):

\[ R_2^{-1}(t) = \lambda_1^{-1}[R_2^{-1}(t - 1) - k_2(t)x_2^T(t)R_2^{-1}(t - 1)] \]

\[ R_1^{-1}(t) = \lambda_2^{-1}[R_1^{-1}(t - 1) - k_1(t)x_1^T(t)R_1^{-1}(t - 1)] \]
RLS algorithm based on the nearest Kronecker product decomposition

Computational complexities of the regular RLS and RLS-NKP algorithms

$L = 500$, with $L_1 = 25$ and $L_2 = 20$
Simulation Results

• conditions
  - $h$ – echo paths from G168 Recommendation, random impulse responses ($L = 500$), and an acoustic echo path ($L = 1024$)
  - input signals – AR1(0.9) process/ speech sequence
  - additive noise $w(t)$ – WGN (SNR=20 dB)
  - measures of performance: normalized misalignment (NM)
    \[
    \text{NM[dB]} = 20 \log_{10} \frac{\|h - \hat{h}(t)\|_2}{\|h\|_2}
    \]

• algorithms
  - proposed RLS algorithm based on the nearest Kronecker product decomposition – RLS-NKP ($\lambda_1 = 1 - 1/[K(PL_1)]$, $\lambda_2 = 1 - 1/[K(PL_2)]$)
  - regular RLS [$\lambda = 1 - 1/(KL)$, with $K > 1$]
Fig. 1. Impulse responses used in the experiments: (a) $L = 500, \xi_{12} = 0.8957$, (b) $L = 500, \xi_{12} = 0.8080$, (c) $L = 500, \xi_{12} = 0.7549$, (d) $L = 500, \xi_{12} = 0.6867$, and (e) $L = 1024, \xi_{12} = 0.6880$. 
Fig. 2. Singular values (normalized with respect to the maximum one) of the matrix $\mathbf{H}$ for the corresponding impulse responses from Fig.1. The size of matrix $\mathbf{H}$ is $L_1 \times L_2$. (a)-(d) $L_1 = 25$ and $L_2 = 20$; (e) $L_1 = L_2 = 32$. 
Fig. 3. Normalized misalignment of the regular RLS and RLS-DCD algorithms ($L = 500$), and RLS-NKP algorithm (using $L_1 = 25$, $L_2 = 20$, and $P < L_2$), for the identification of the impulse responses from Figs. 1(a) and (b). The input signal is an AR(1) process and the impulse response changes at times 3 and 6 seconds.
Fig. 4. Normalized misalignment of the regular RLS and RLS-DCD algorithms ($L = 500$), and RLS-NKP algorithm (using $L_1 = 25, L_2 = 20$, and $P < L_2$), for the identification of the impulse responses from Figs. 1(c) and (d). The input signal is an AR(1) process and the impulse response changes at times 3 and 6 seconds.
Fig. 5. Normalized misalignment of the RLS algorithm \((L = 1024)\) and RLS-NKP algorithm (using \(L_1 = L_2 = 32\), and \(P < L_2\)), for the identification of the impulse responses from Figs. 1(e). The input signal is a speech sequence and the impulse response changes at time 5 seconds.
Conclusions and Perspectives

• We have proposed the RLS-NKP algorithm.

• Suitable for the identification of low-rank models, like the echo paths.

• The tracking capabilities of the RLS-NKP algorithm are better as compared to the conventional RLS algorithm.

• The computational complexity of the proposed algorithm could be much lower as compared to the RLS.


Thank you for your attention!