Solving Complex Quadratic Equations with Full-rank Random Gaussian Matrices

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1. Problem formulation

We minimize the following objective function $f(z)$:

$$f(z) = \frac{1}{m} \sum_{i=1}^{m} |z^T A_i z - y_i|^2,$$

(2)

using gradient descent:

$$z^{(t+1)} = z^{(t)} - \eta \nabla f(z),$$

(3)

where $\eta > 0$ is the step size.

2. System of quadratic equations

- Quadratic measurements obtained with high-rank measurement matrices arise in applications such as unassigned distance geometry problem.
- Most prior works focus on rank-1 psd measurement matrices or real measurements.
- Measurement Model:

$$y_i = x^T A_i x, \quad i = 1, \cdots, m.$$ (1)

- $x \in \mathbb{C}^n$ is the complex signal.
- $y_i \in \mathbb{C}$ is the i-th complex quadratic measurement.
- $A_i \in \mathbb{C}^{n \times n}$ is the i-th complex random Gaussian measurement matrix.

3. Main theorem

When $m \geq Cn$ for some sufficiently large constant $C$,

1. There exists a choice of $1 > \nu > 0$, $1 > \rho > 0$, $\alpha > 0$, $\beta > 0$ such that $RC(\alpha, \beta, \rho)$ holds on $E(\rho)$ with high probability.
2. Furthermore, if the step size $0 < \eta \leq \frac{2}{\eta}$, the gradient descent with the spectral initializer $z^{(0)}$ converges linearly to $x$

$$\text{dist}(z^{(t)}, x) \leq \left(1 - \frac{2\beta}{\alpha}\right) \rho \cdot \|x\|_2,$$

(11)

with high probability.

4. Lemma

When $m \geq Cn$ for some sufficiently large constant $C$, for all $p, q \in \mathbb{C}^n$

satisfying $\|p\|_2 = 1$, $\|q\|_2 = 1$ and every $\nu > 0$, the following

$$\left\| \frac{1}{m} \sum_{i=1}^{m} p^T A_i^T q \cdot A_i \cdot 2qp \right\|_2 \leq \nu,$$

(8)

holds with high probability.

5. Spectral initialization

The left or right singular vector $z^{(0)}$ of the following matrix:

$$S = \frac{1}{m} \sum_{i=1}^{m} \pi_i A_i,$$

(5)

where $\eta > 0$.

- For sufficiently large $m$, with high probability $S$ concentrates around its expectation in terms of spectral norm.

$$E[S] = 2xx^T$$

(6)

- The spectral initializer $z^{(0)}$ is close to a global optimum $x$ [1,2].

$$\text{dist}(z^{(0)}, x) \leq \delta \|x\|_2$$

(7)

holds with high probability.

6. Experimental results

Experimental results show the effectiveness of the proposed algorithm.

- Figure 2: Left: $\text{dist}(z^{(t)}, x)$; Right: Phase transition of the success rate.

- Figure 3: Recovery of the UIUC logo.

References


Figure 1: A good initialization is needed to solve a nonconvex optimization problem via gradient descent.

Figure 2: Left: $\text{dist}(z^{(t)}, x)$; Right: Phase transition of the success rate.

Figure 3: Recovery of the UIUC logo.