Resolution Limits of 20 Questions Search Strategies for Moving Targets

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Origin: 20 Questions Game

Responder

Questioner
Origin: 20 Questions Game

Q1: Is it edible?
A1: Yes!
Origin: 20 Questions Game

Q1: Is it edible?
A1: Yes!

Q2: Is it a fruit?
A2: Yes!
Origin: 20 Questions Game

Q1: Is it edible?
   A1: Yes!

Q2: Is it a fruit?
   A2: Yes!

... 

Q20: ...
   A20: ...

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Q1: Is it edible?
A1: Yes!

Q2: Is it a fruit?
A2: Yes!

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Q20: ...
A20: ...

Banana
Ulam-Rényi game: a noisy channel is introduced to model the behavior of the responder who can lie to decline to answer queries
Recap: Search with Noise for a Stationary Target

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- Target: estimate a random variable $S$ with unknown distribution.
Recap: Search with Noise for a Stationary Target

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- Task: design queries and decoder (scheme/strategy/procedure)
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Motivation: diverse applications including:
- medical diagnosis, chemical triage, human-in-the-loop decision-making
- fault-tolerant communications, beamforming design in millimeter wave communication
- target localization with a sensor network, object localization in an image
Adaptive and Non-Adaptive Query Schemes

- A query asks whether $S$ lies in a certain set $A \subset [0, 1]$
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- Query schemes can be classified as adaptive and non-adaptive
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Adaptive

-Responder

Noisy Channel

Decoder

Stop

$\hat{S}$
Adaptive and Non-Adaptive Query Schemes

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### Adaptive

- Respond \( S \)
- Noisy Channel \( X_i \)
- Decoder \( Y_i \)
- Stop \( \hat{S} \)

### Non-adaptive

- Respond \( S \)
- Noisy Channel \( X_1, X_2, \ldots, X_n \)
- Decoder \( Y_1, Y_2, \ldots, Y_n \)
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Query schemes can be classified as adaptive and non-adaptive

Adaptive

Noisy Channel

Decoder

Responder

Non-adaptive

Noisy Channel

Decoder

Responder

A. O. Hero (University of Michigan, Ann Arbor)

20 Questions for Moving Target Search

Paper 1204, ICASSP 2021
Given a query (measurement) $A$, a responder’s noiseless answer is corrupted by measurement-dependent noise via $P_{Y|X}^{A}$. 

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Measurement-Dependent\textsuperscript{1} Noise Model

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The channel $P_{Y|X}^A$ depends on the query $\mathcal{A}$ only through a bounded Lipschitz continuous function $f : [0, 1] \rightarrow \mathcal{R}$ of its size $|\mathcal{A}|$.

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- For any $(q_1, q_2) \in [0, 1]^2$, $|f(q_1) - f(q_2)| \leq \mu |q_1 - q_2|$.
- When $f$ is a constant value function, the noise model reduces to a measurement-independent model.

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Problem Formulation: Search for a Multidimensional Moving Target

Target: estimate the trajectory of a $d$-dimensional moving target with initial location $\mathbf{S} = (S_1, \ldots, S_d) \in [0, 1]^2$ and moving velocity $\mathbf{V} = [V_1, \ldots, V_d] \in [-v_+, v_+]^d$
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Applications: search for a moving target (e.g., a car, wild animals, missing airplane) using sensor networks or satellites
The Torus model for the Moving Target

Given initial location \( s = (s_1, \ldots, s_d) \) and moving velocity \( v = (v_1, \ldots, v_d) \), at each time \( t \in \mathbb{R}_+ \), the real time location of the target at \( i \)-th dimension satisfies:

\[
1(s_i, v_i, t) := \begin{cases} 
1 & \text{if } \text{mod}(s_i + tv_i, 2) = 1, \\
 s_i + tv_i - \lfloor s_i + tv_i \rfloor & \text{if } s_i + tv_i \in \bigcup_{h \in \mathbb{N}} [2h, 2h+1), \\
 \lfloor s_i + tv_i \rfloor - (s_i + tv_i) & \text{otherwise,}
\end{cases}
\]
Definition of Non-Adaptive Query Procedures

Given any \((n, d) \in \mathbb{N}^2, \delta \in \mathbb{R}_+\) and \(\varepsilon \in [0, 1)\), a \((n, d, \delta, \varepsilon)\)-non-adaptive query procedure consists of

- \(n\) queries \(A^n\) where at time \(i\), questioner asks whether the moving target’s current location lies in set \(A_i \subset [0, 1]^d\)
- and a decoder \(g : \mathcal{Y}^n \rightarrow [0, 1]^d \times \mathcal{V}^d\) such that the worst-case excess-resolution probability satisfies

\[
P_e(n, d, \delta) := \sup_{i_{sv}} \Pr \left\{ \max_{t \in [0:n]} \| l(\hat{S}, \hat{V}, t) - l(S, V, t) \|_{\infty} > \delta \right\} \leq \varepsilon.
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- Accurate estimation of the trajectory implies accurate estimation of \((S, V)\) and vice versa
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  - Excess-resolution event won’t occur if \(|\hat{S}_i - S_i| \leq \alpha \delta\) and \(n|\hat{V}_i - V_i| \leq (1 - \alpha)\delta\) for all dimensions \(i \in [d]\)
Explanation of the Non-Excess Resolution Event

For each $i \in [d]$, the $i$-th dimension does \textit{not} incur excess-resolution if the estimated trajectories are within $\delta$ around the true trajectory at each time (in green shaded region).
Excess-Resolution Case 1: Wrong Estimate of Initial Location

If the initial location $s_i$ is estimated wrongly such that $|\hat{s}_i - s_i| > \delta$, then an excess-resolution event occurs.
If the velocity $v_i$ is estimated wrongly such that $n|\hat{v}_i - v_i| > 2\delta$, then an excess-resolution event occurs.
Excess-Resolution Case 3: Wrong Estimate of Both

If both the location $s_i$ and the velocity $v_i$ are estimated wrongly, an excess-resolution event occurs.
Longer search time $n$ requires more accurate estimation of the initial location and the velocity
Given any number of queries $n \in \mathbb{N}$ and $\varepsilon \in [0, 1]$,

$$
\delta^*(n, d, \varepsilon) := \inf\{\delta \in \mathbb{R}_+ : \exists \text{ an } (n, d, \delta, \varepsilon)-\text{non-} \text{adaptive query procedure}\}.
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Fundamental Limit

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- **minimal resolution** achievable by any non-adaptive query procedure with $n$ queries and excess-resolution probability $\varepsilon$
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- Dual quantity (sample complexity):

  $$n^*(d, \delta, \varepsilon) = \inf\{n \in \mathbb{N}: \delta^*(n, d, \varepsilon) \leq \delta\}$$
Characterization of the Minimal Achievable Resolution $\delta^*(n, d, \varepsilon)$

**Theorem 1**

For any $\varepsilon \in (0, 1)$ and $d \in \mathbb{N}$, the minimal achievable resolution $\delta^*(n, d, \varepsilon)$ satisfies:

- If $nv_+ = O(n^t)$ for $t \in [0.5, 1)$,
  
  $$-2d \log \delta^*(n, d, \varepsilon) = nC + O(nv_+);$$

- If $nv_+ = O(n^t)$ for $t \in [0, 0.5)$
  
  $$-2d \log \delta^*(n, d, \varepsilon) = nC + \sqrt{nV_\varepsilon} \Phi^{-1}(\varepsilon) + O(\max\{nv_+, \log n\});$$
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- The two assumptions on maximal speed $v_+$ specify two regimes
  - Regime 1: # queries is greater than $O(1/v_+^2)$ (fast target, $v_+ = O(\frac{1}{\sqrt{n}})$ and $v_+ = o(1)$)
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Theorem 1

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  - Regime 2: # queries is fewer than \( O(1/v_+^2) \) (slow target, \( v_+ = o(\frac{1}{\sqrt{n}}) \))
- In Regime 2, the first-order asymptotic result is not sufficient (see next slide)
First- and Second-order Asymptotics for the Resolution Decay Rate

- First-order asymptotics: the asymptotic resolution decay rate

\[
\lim_{n \to \infty} \frac{-\log \delta^*(n, d, \varepsilon)}{n} = \frac{C}{2d}
\]
First- and Second-order Asymptotics for the Resolution Decay Rate

- **First-order asymptotics:** the asymptotic resolution decay rate

\[
\lim_{n \to \infty} \frac{- \log \delta^*(n, d, \varepsilon)}{n} = \frac{C}{2d}
\]

- **Second-order asymptotics:** characterize the backoff from first-order \((nv_+ = o(\sqrt{n}))\)
Further Remarks

- Refines the result by Kaspi et al., TIT 2018 (Theorem 3):
  - Second-order asymptotic, non-vanishing vs first-order asymptotic, vanishing
  - Any measurement dependent channel vs a measurement dependent BSC
  - Multidimensional vs one-dimensional
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- Consistent with intuition that searching for a moving $d$-dimensional target is roughly equivalent to searching for a $2d$-dimensional target
  - Analysis is totally different: account for all trajectories and twist of location and velocity
  - Much more complicated: time complexity is $O(n^{2d+1} \nu_d M^{2d})$ to search for a moving target v.s. $O(nM^{2d})$ to search for a stationary target when the target resolution is $\frac{1}{M}$
An Important Implication: Phase Transition

- Minimal excess-resolution probability $\varepsilon^*(n, d, \delta)$ when $nv_+ = o(\sqrt{n})$

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\varepsilon^*(n, d, \delta) = \Phi \left( \frac{-d \log \delta - nC}{\sqrt{nV_\varepsilon}} \right) + o(1)
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- BSC with crossover probability \((2|A| + 0.5) \times 0.2\)
The Impact of the Maximal Speed for A One-Dimensional Target

- Consider uniformly distributed location $S \in [0, 1]$ and velocity $V \in [-v_+, v_+]$
- Consider BSC with crossover probability $(|A| + 0.5) \times 0.05$ and set $\varepsilon = 0.1$
- Gaussian approximation (Second-order asymptotic approximation)

\[ v_+ = \frac{1}{n} \quad \text{and} \quad v_+ = \frac{\log n}{n} \]
Numerical Simulation

- $n\nu_+ = 0.1$
- $10^4$ independent trials for each $n$
Reflections and Future Research Directions

- Searching for a moving target with unknown initial location and velocity
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  - Second-order asymptotic approximation to the minimal achievable resolution for “slow” moving target

Future research directions
- Generalize the torus model for practical uses
- Low complexity practical algorithms which achieve derived benchmarks
- Adaptive query procedure (Benefit of adaptivity?)
- Simultaneous search for multiple moving targets
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  - Searching for a $d$-dimensional moving target takes much longer than searching for a $2^d$-dimensional stationary target

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