Optimizing Spectral Diversity for Graph Signal Coarsening

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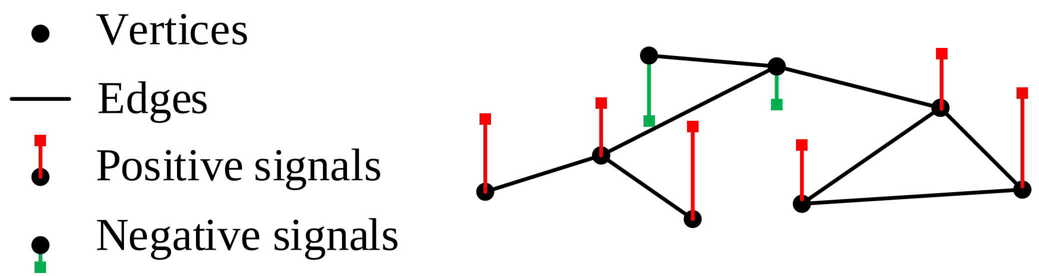
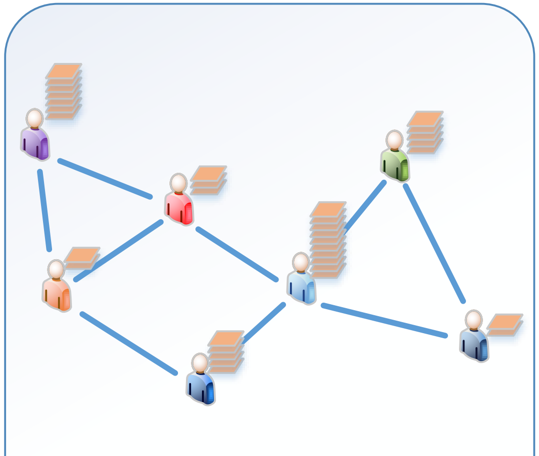
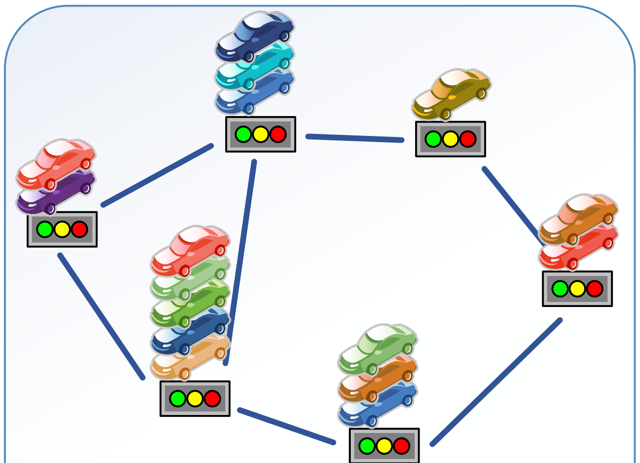
Signals on graphs

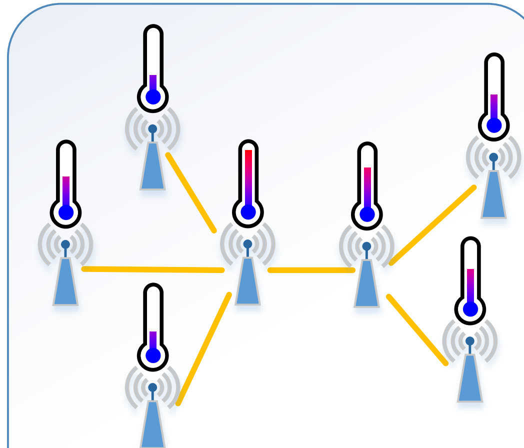




Numbers of documents on co-author network

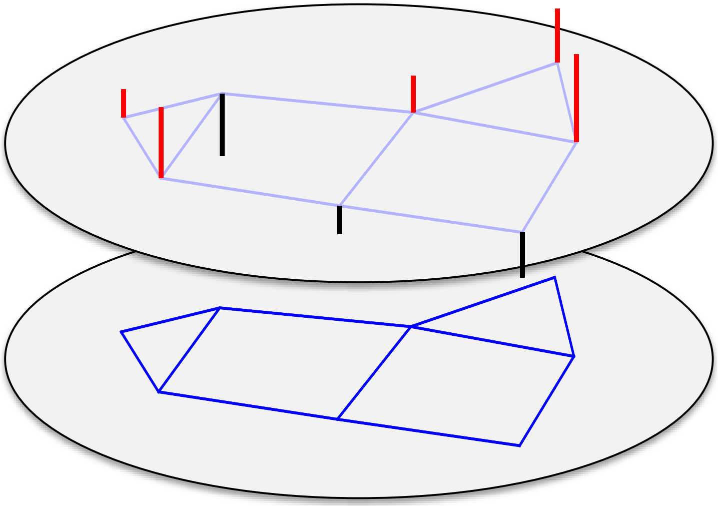


Traffic queue length on road network



Temperatures on sensor network

Signals on graphs



Associate each vertex
with a scalar value
to form a graph signal

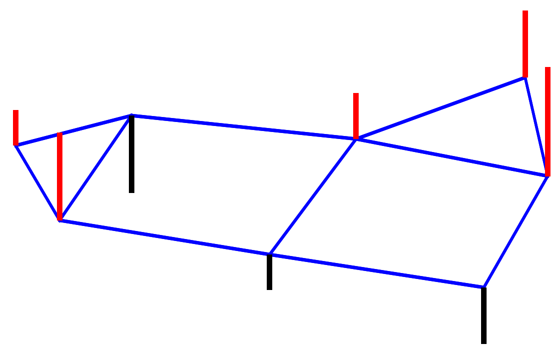
$$\mathbf{W} = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

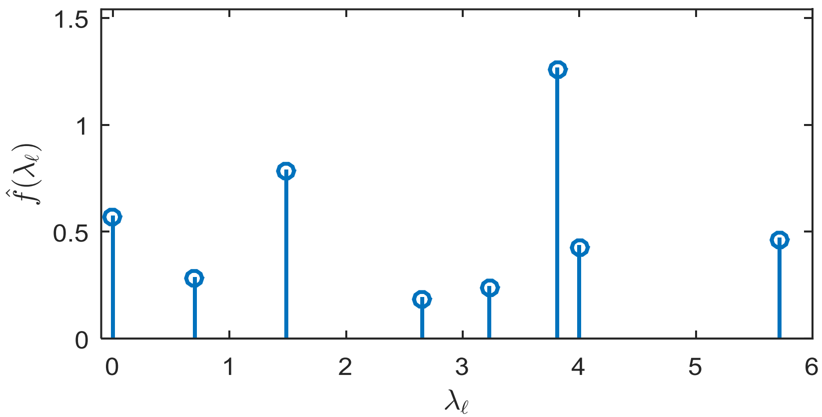
Graph: Weight Matrix

$$\mathbf{f} = \begin{pmatrix} 0.3 \\ 1 \\ -1 \\ -0.3 \\ -0.6 \\ 0.5 \\ 1.2 \\ 1 \end{pmatrix}$$

Signal: Vector

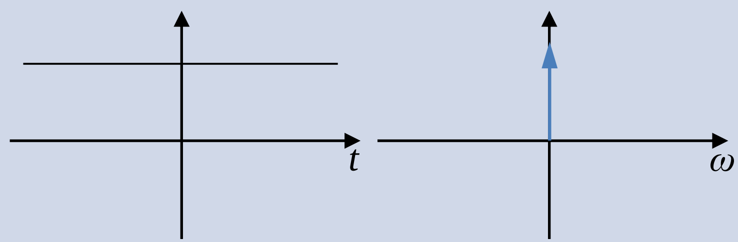
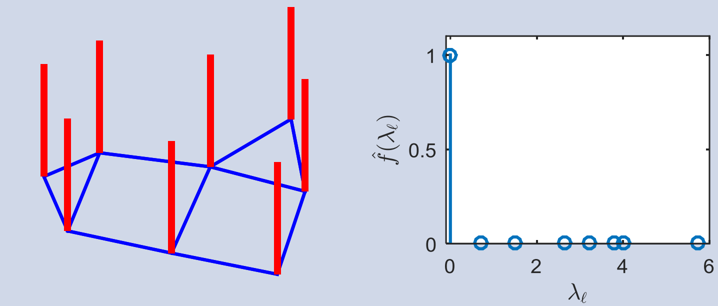
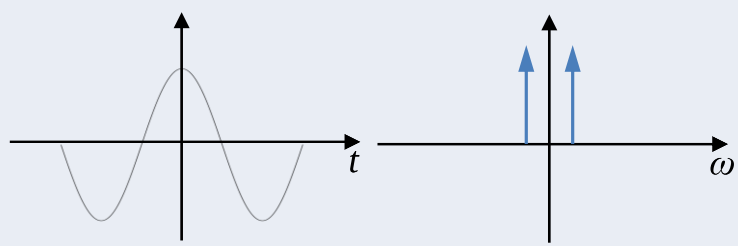
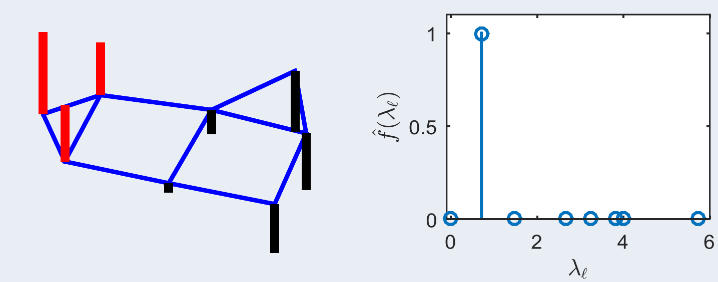
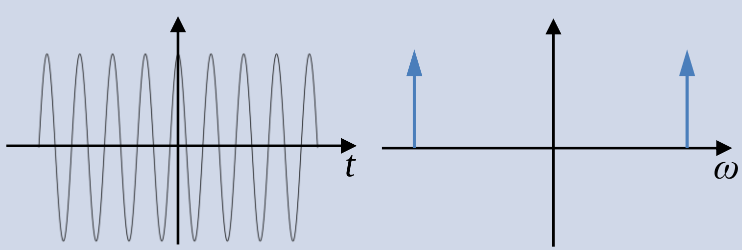
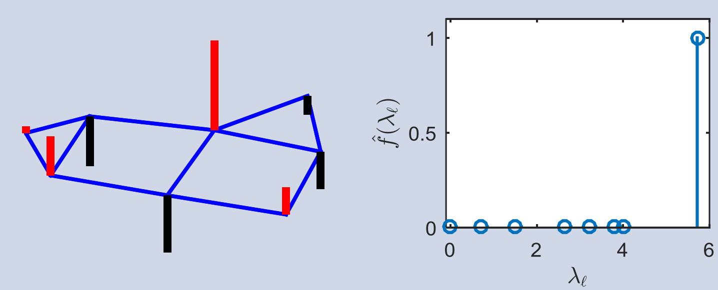
Vertex domain and spectral domain representations based on Laplacian





Vertex domain Representation	$(\mathcal{G}, \mathbf{f})$		Spectral domain Representation	$(\boldsymbol{\lambda}, \hat{\mathbf{f}})$
Graph Laplacian	$\mathbf{L} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^{\mathrm{T}}$	$\Longleftrightarrow$	$\boldsymbol{\lambda} = \mathrm{diag}(\boldsymbol{\Lambda})$	Graph spectrum
Signal	$\mathbf{f}$	$\Longleftrightarrow$	$\hat{\mathbf{f}} = \mathbf{U}^{\mathrm{T}}\mathbf{f}$	Signal Spectrum

Single-frequency signals: Classical vs. Graph

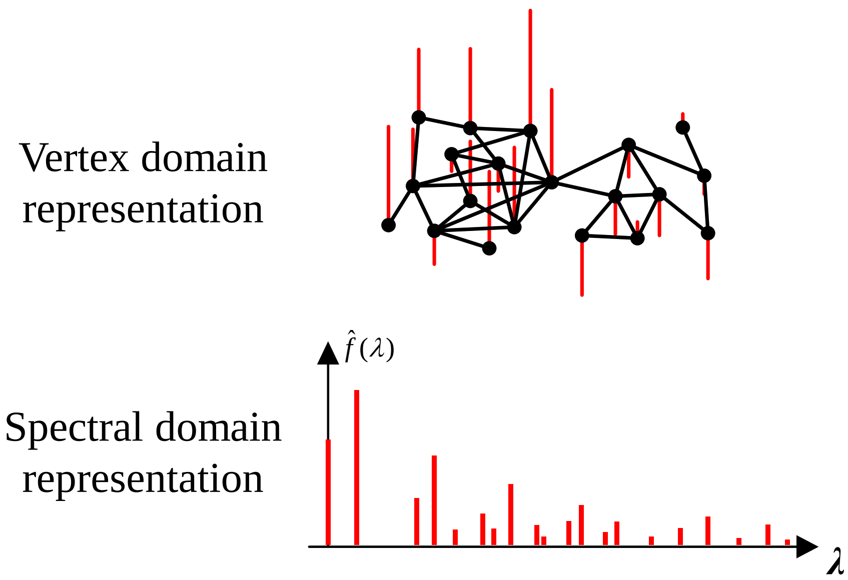
	Classical signal processing	Signal processing on graphs
DC signal		
Low-frequency signal		
High-frequency signal		

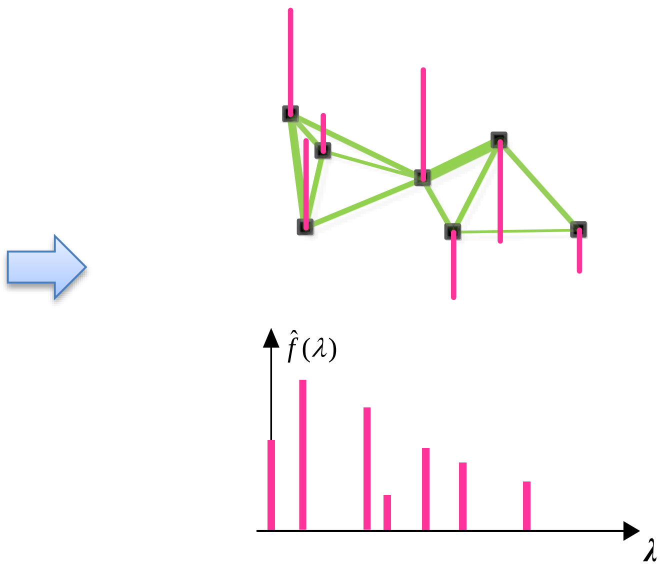
Graph signal coarsening

- **Joint dimensionality reduction** for graph and signal with vertex domain and spectral domain similarities

Given: Original graph signal $(\mathcal{G}^{\text{in}}, \mathbf{f}^{\text{in}})$

Find: Coarsened graph signal $(\mathcal{G}, \mathbf{f})$ with n vertices





Graph signal coarsening

- Joint dimensionality reduction for graph and signal with vertex domain and spectral domain similarities

Given: Original graph signal $(\mathcal{G}^{\text{in}}, \mathbf{f}^{\text{in}})$

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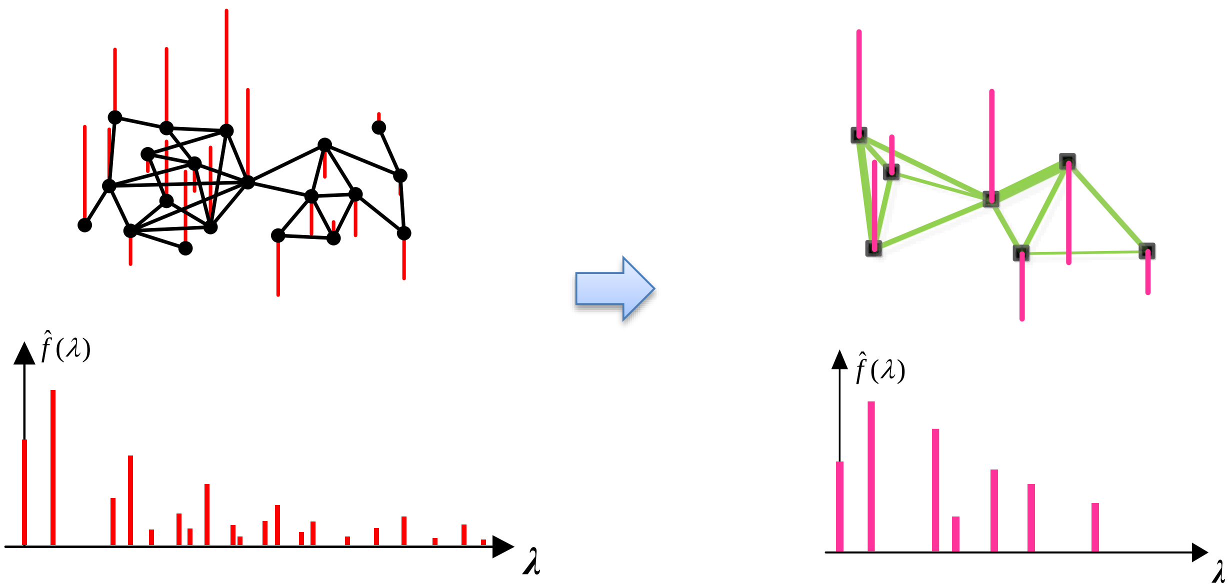
Vertex domain representation

Spectral domain representation



- Question: How to evaluate the quality of graph signal coarsening?

Spectral Diversity

- Objective:
  - Measure the similarity of variation speeds for signals across the graphs
- Assumption
  - Single frequency signals with the same frequency have the same variation speeds, even for signals reside on different graphs

Spectral Diversity

- Spectral Energy Cumulative Distribution Function (SECDF)

$$F(\lambda) = \frac{\sum_{i=1}^{N} \hat{f}_i^2 \cdot 1_{\lambda_i \le \lambda}}{\sum_{i=1}^{N} \hat{f}_i^2} \qquad \text{where} \qquad 1_{\lambda_i \le \lambda} = \begin{cases} 1, & \lambda_i \le \lambda; \\ 0, & \lambda_i > \lambda. \end{cases}$$

  - Cumulative distribution function of signal energy in the spectral domain

- Spectral Diversity

$$\mathrm{D}[(\mathcal{G}_1, \mathbf{f}_1), (\mathcal{G}_2, \mathbf{f}_2)] = \int_0^{+\infty} |\, F_1(\lambda) - F_2(\lambda)|\, d\lambda$$

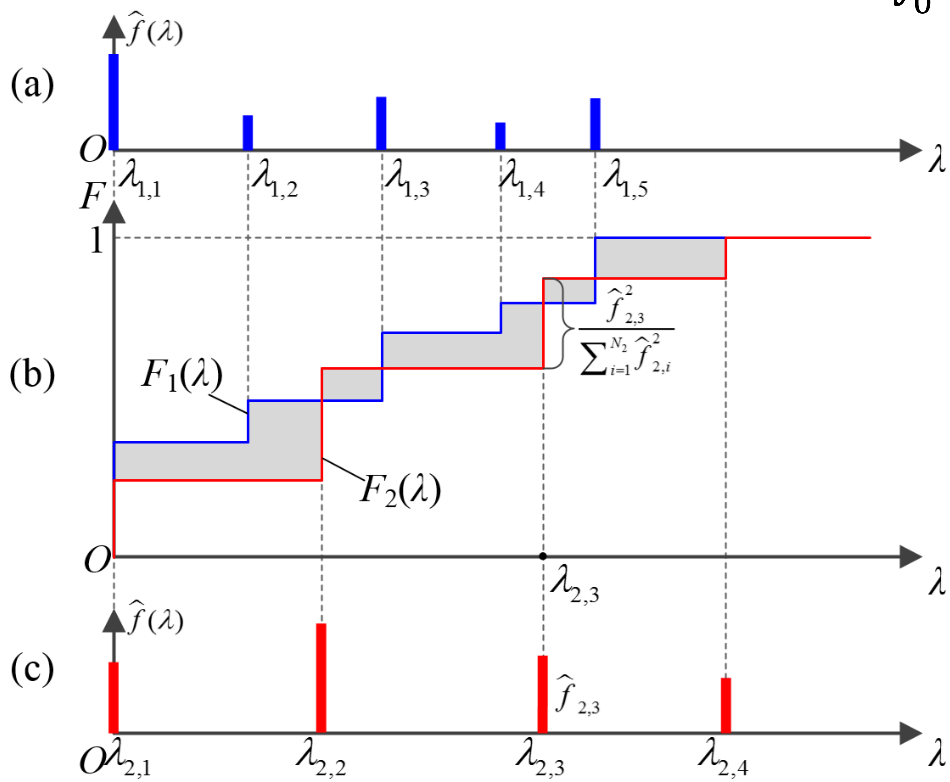
Spectral Diversity

- SECDF

$$F(\lambda) = \frac{\sum_{i=1}^{N} \hat{f}_i^2 \cdot 1_{\lambda_i \le \lambda}}{\sum_{i=1}^{N} \hat{f}_i^2} \quad \text{where} \quad 1_{\lambda_i \le \lambda} = \begin{cases} 1, & \lambda_i \le \lambda; \\ 0, & \lambda_i > \lambda. \end{cases}$$

- Spectral Diversity

$$\mathrm{D}[(\mathcal{G}_1, \mathbf{f}_1), (\mathcal{G}_2, \mathbf{f}_2)] = \int_0^{+\infty} |F_1(\lambda) - F_2(\lambda)| d\lambda$$



SECDF: $F_1(\lambda), F_2(\lambda)$

Spectral diversity:

Area of shaded region

Optimizing Spectral Diversity for Graph Signal Coarsening

- This problem can be formulated into

$$\min_{\mathcal{G},\mathbf{f}} \mathrm{D}[(\mathcal{G},\mathbf{f}),(\mathcal{G}^{\mathrm{in}},\mathbf{f}^{\mathrm{in}})], \text{ subject to } |\mathcal{V}| = n,$$

Coarsened graph signal

Original graph signal

Expected vertex count

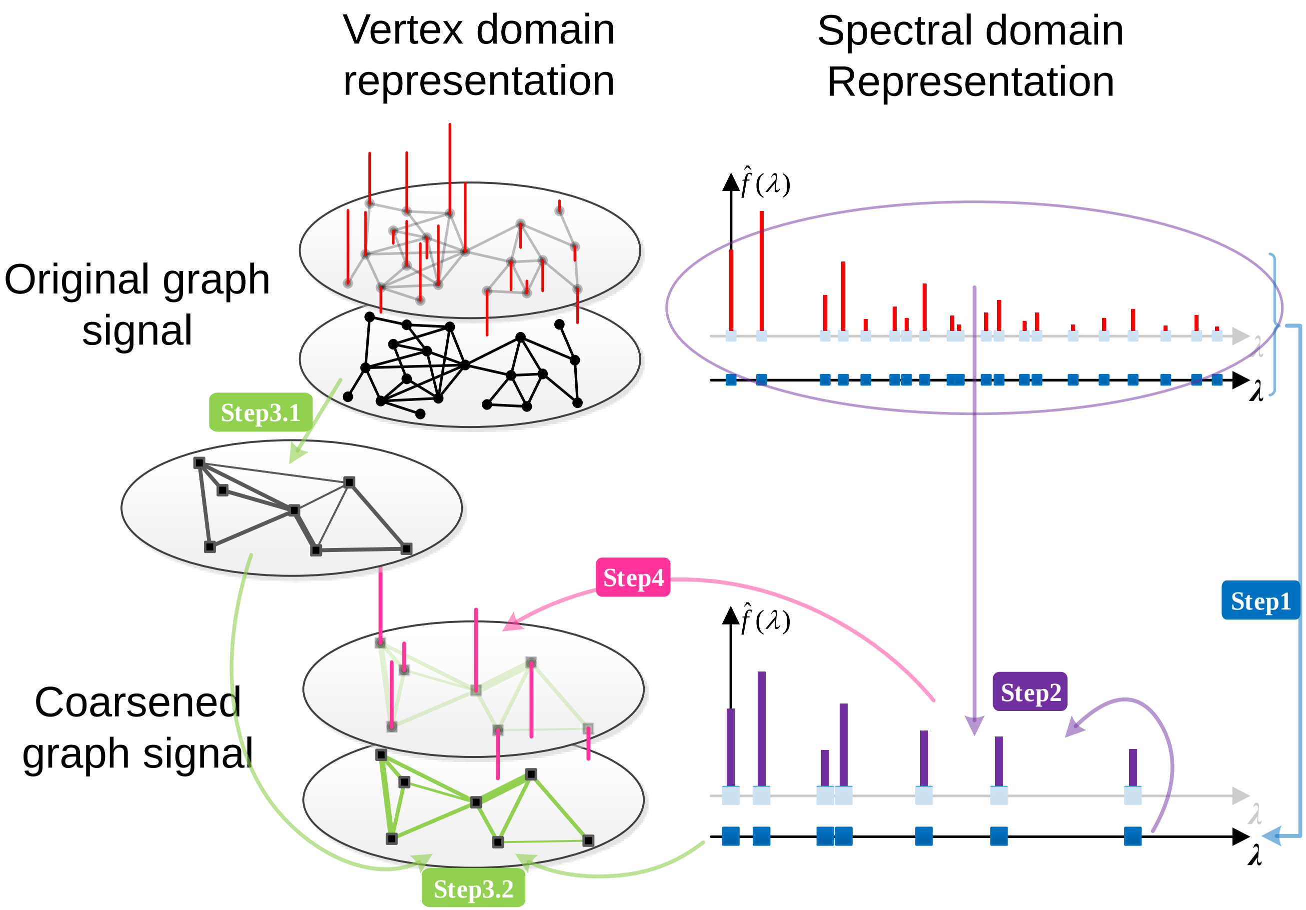
Optimizing Spectral Diversity for Graph Signal Coarsening

- This problem can be formulated into

$$\min_{\mathcal{G},\mathbf{f}} \mathrm{D}[(\mathcal{G},\mathbf{f}),(\mathcal{G}^{\mathrm{in}},\mathbf{f}^{\mathrm{in}})], \text{ subject to } |\mathcal{V}| = n,$$

- It can be split into two subproblems
  - Obtain the spectra of the coarsened graph and coarsened signal
  - Get the graph signal satisfying expected spectra

Optimizing Spectral Diversity for Graph Signal Coarsening



① Obtain the spectrum of the new graph with greedy method

② Get the spectrum of the coarsened signal with spectral bin method

③ Construct the coarsened graph with ADMM

④ Acquire the coarsened signal with inverse Fournier transform on graphs

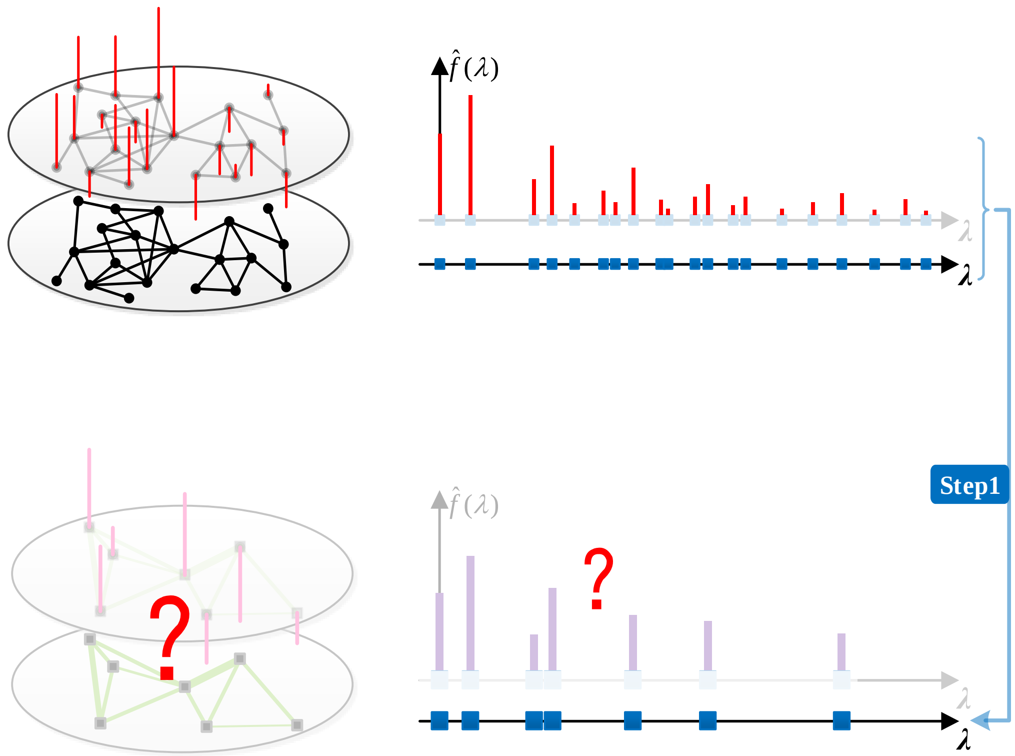
Optimizing Spectral Diversity for Graph Signal Coarsening

Vertex domain representation

Spectral domain Representation

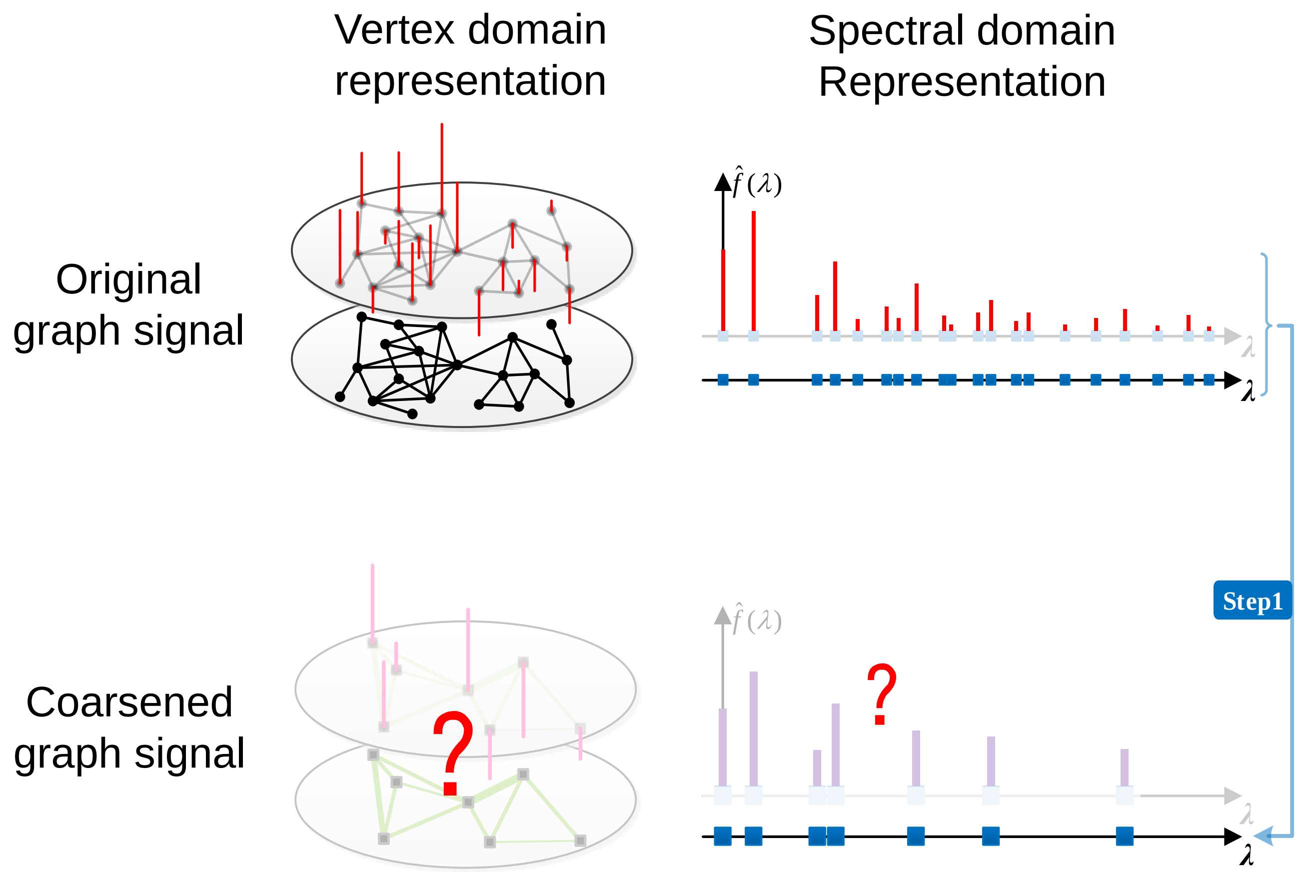
Original graph signal

Coarsened graph signal



① Obtain the spectrum of the new graph with greedy method

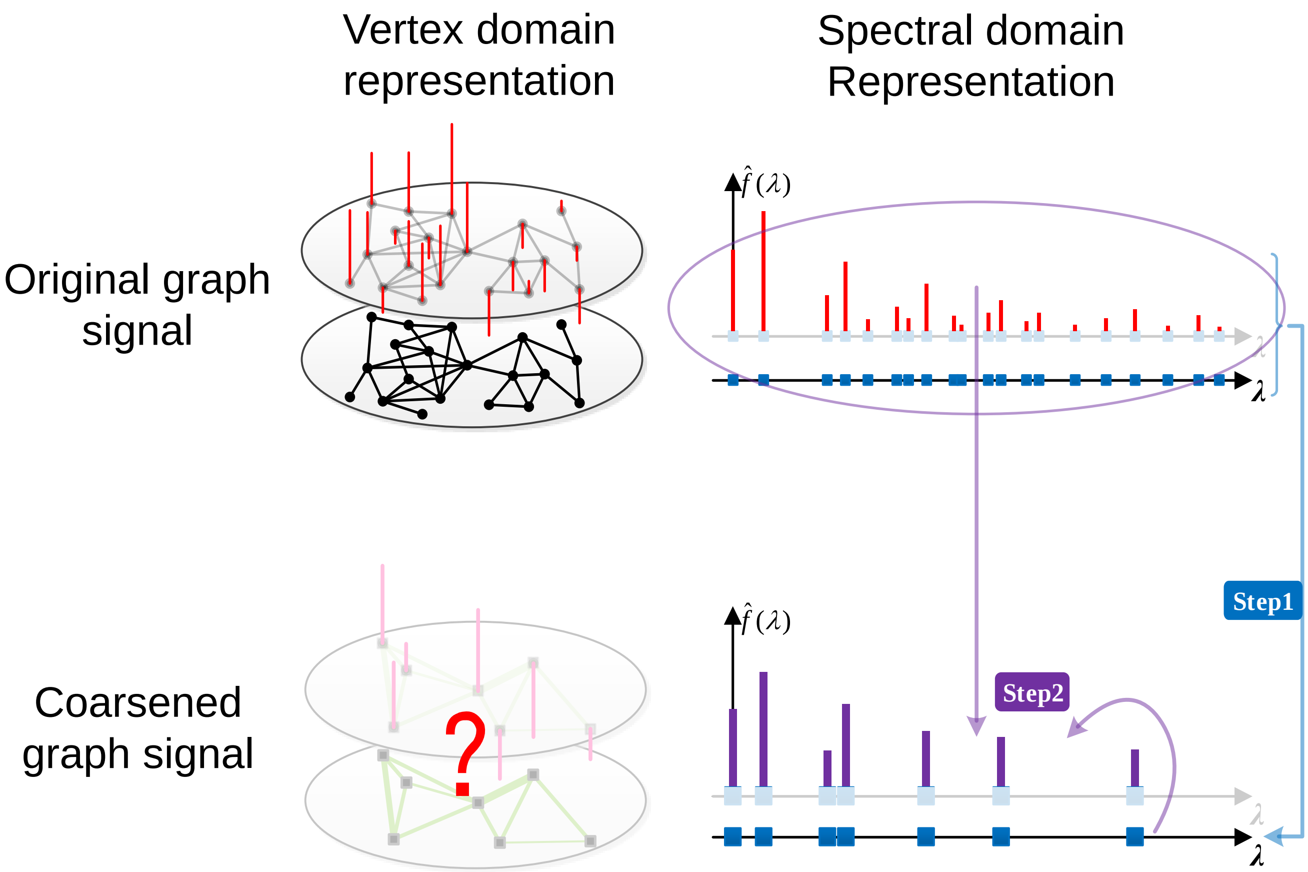
Optimizing Spectral Diversity for Graph Signal Coarsening



① Obtain the spectrum of the new graph with greedy method

- The new graph spectrum should be a subset of the original one

Optimizing Spectral Diversity for Graph Signal Coarsening

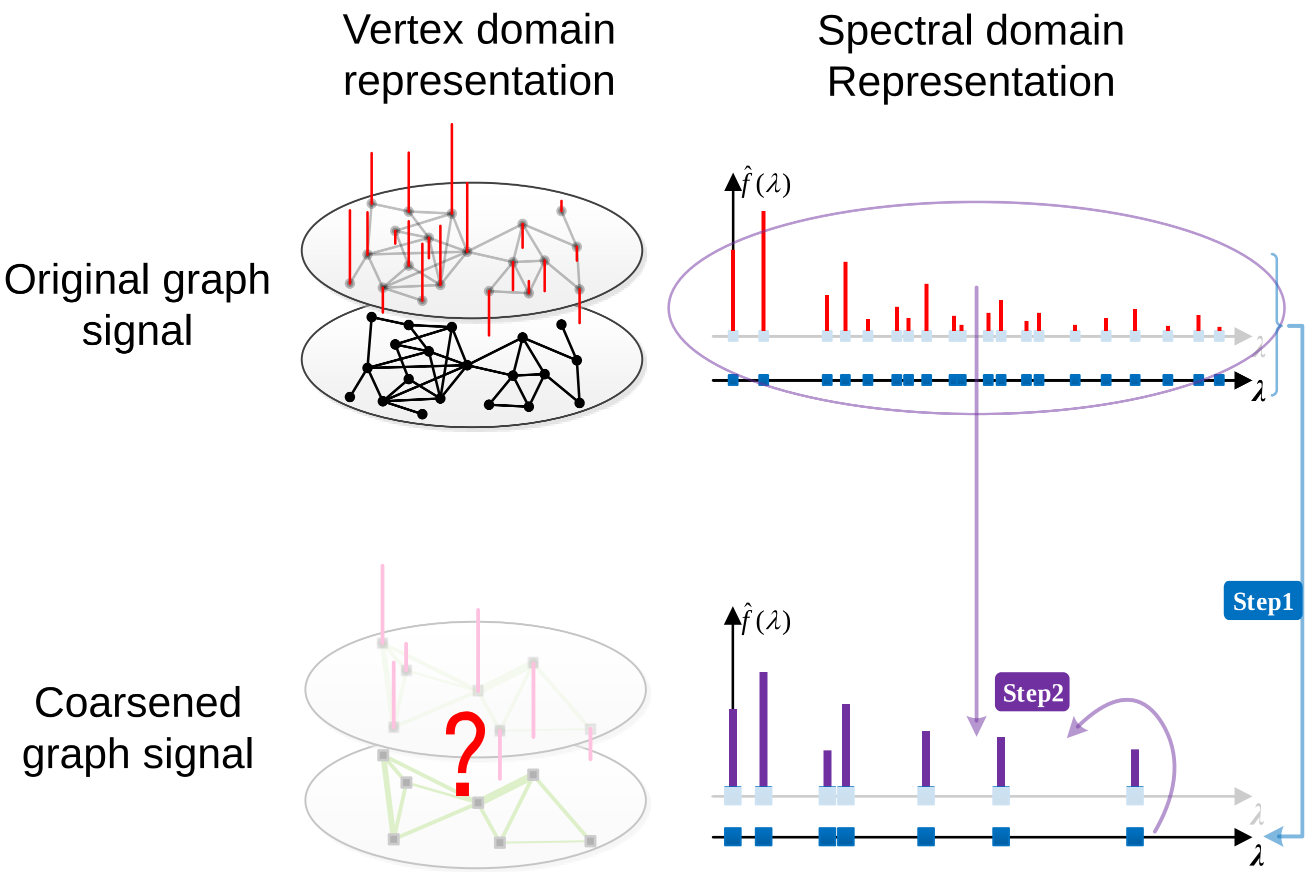


① Obtain the spectrum of the new graph with greedy method

② Get the spectrum of the coarsened signal with spectral bin method*

*Method proposed in P. Liu, X. Wang, and Y. Gu, Graph Signal Coarsening: Dimensionality Reduction in Irregular Domain, IEEE Global Conference on Signal and Information Processing (GlobalSIP), 966-970, Dec. 3-5, 2014, Atlanta, Georgia, USA.

Optimizing Spectral Diversity for Graph Signal Coarsening

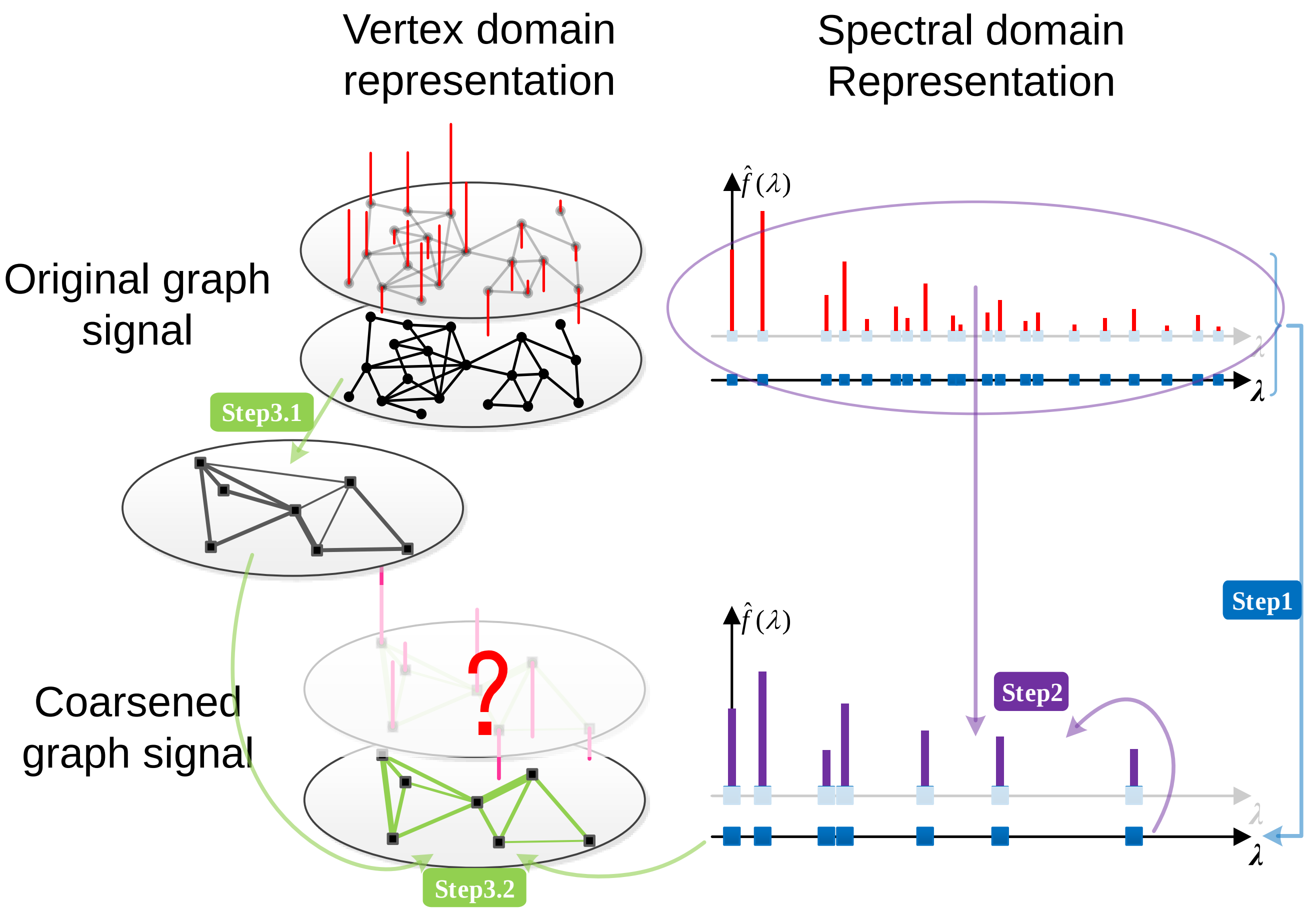


① Obtain the spectrum of the new graph with greedy method

② Get the spectrum of the coarsened signal with spectral bin method*

- Spectral bin method always gets the optimal solution for such problem

Optimizing Spectral Diversity for Graph Signal Coarsening



① Obtain the spectrum of the new graph with greedy method

② Get the spectrum of the coarsened signal with spectral bin method

③ Construct the coarsened graph with ADMM

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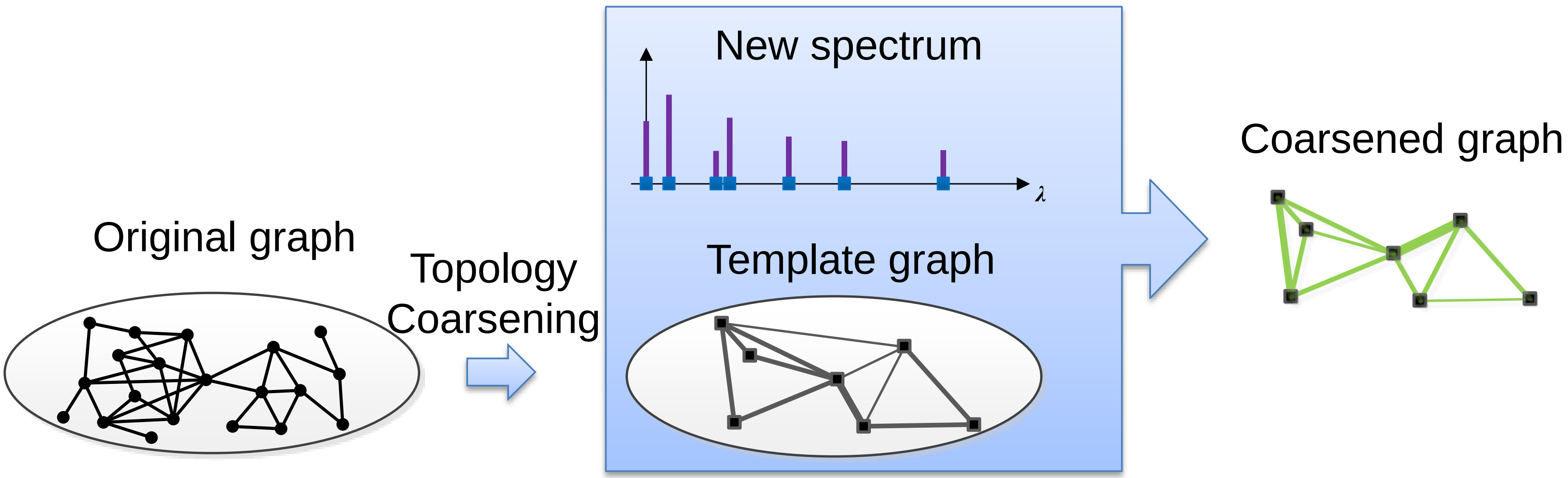
- Constraint: the new graph should satisfy the new spectrum
- Objective: Ensure the similarity of the coarsened graph and a template graph in the vertex domain. The template graph is obtained with topology coarsening method



③ Construct the coarsened graph with ADMM

- Laplacian of the original graph and the template graph are $\mathbf{L}^{\mathrm{in}}$ and $\tilde{\mathbf{L}}$, respectively
- Laplacian of the new graph is $\mathbf{L} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^{\mathrm{T}}$, where $\boldsymbol{\Lambda} = \mathrm{diag}(\boldsymbol{\lambda})$, $\boldsymbol{\lambda}$ is the expected graph spectrum

- The problem can be formulated into:

$$\min F(\mathbf{U}, t) = \|\mathbf{L} - t\tilde{\mathbf{L}}\|_{\mathrm{F}}^2 + \tau\|\mathbf{L}\|_1$$

subject to

$$\mathbf{L} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^{\mathrm{T}}$$

$$\mathbf{U} \in \mathcal{O}_n = \{\, \mathbf{X} \in \mathbb{R}^{n\times n} \mid \mathbf{X}\mathbf{X}^{\mathrm{T}} = \mathbf{X}^{\mathrm{T}}\mathbf{X} = \mathbf{I}_n \}$$

$$\mathbf{L} \in \mathcal{M}_1 = \{\mathbf{X} \in \mathbb{R}^{n\times n} \mid \mathbf{X}\mathbf{1}_{n\times 1} = \mathbf{0}_{n\times 1}\}$$

$$\mathbf{L} \in \mathcal{N}_- = \{\, \mathbf{X} \in \mathbb{R}^{n\times n} \mid x_{p,q} \le 0, p \neq q \,\}$$

③ Construct the coarsened graph with ADMM

- Laplacian of the original graph and the template graph are $\mathbf{L}^{\mathrm{in}}$ and $\tilde{\mathbf{L}}$, respectively
- Laplacian of the new graph is $\mathbf{L} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^{\mathrm{T}}$, where $\boldsymbol{\Lambda} = \mathrm{diag}(\boldsymbol{\lambda})$, $\boldsymbol{\lambda}$ is the expected graph spectrum

- The problem can be formulated into:

$$\min F(\mathbf{U}, t) = \underbrace{\boxed{\|\mathbf{L} - t\tilde{\mathbf{L}}\|_{\mathrm{F}}^2}}_{\text{Ensure topological similarity}} + \underbrace{\boxed{\tau\|\mathbf{L}\|_1}}_{\text{Induce sparsity}}$$

subject to

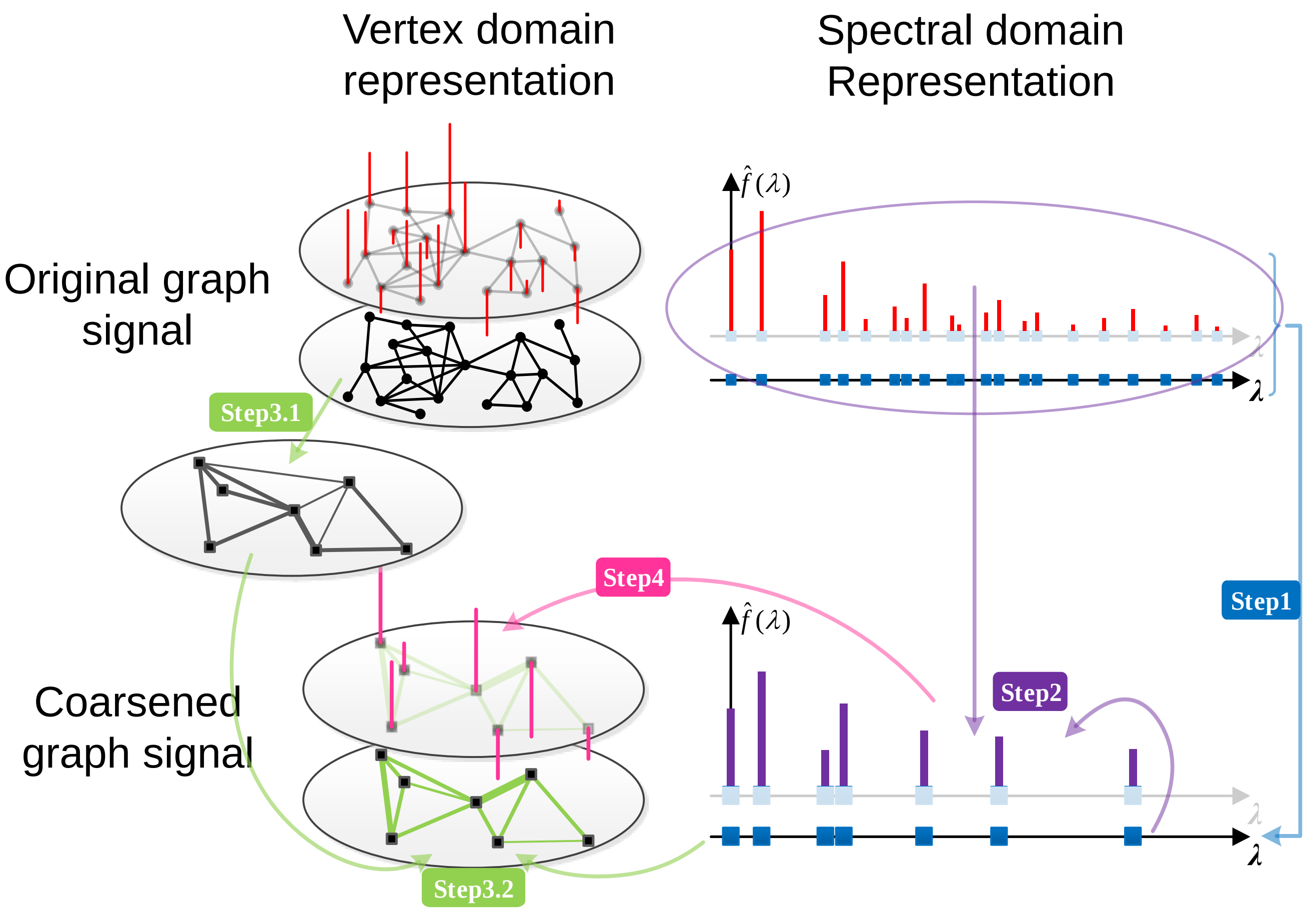
$$\boxed{\mathbf{L} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^{\mathrm{T}}} \quad \text{Graph spectrum constraint}$$

$$\boxed{\mathbf{U} \in \mathcal{O}_n = \{\, \mathbf{X} \in \mathbb{R}^{n\times n} \mid \mathbf{X}\mathbf{X}^{\mathrm{T}} = \mathbf{X}^{\mathrm{T}}\mathbf{X} = \mathbf{I}_n \}} \quad \text{Eigenvalue decomposition}$$

$$\boxed{\begin{array}{l} \mathbf{L} \in \mathcal{M}_1 = \{\mathbf{X} \in \mathbb{R}^{n\times n} \mid \mathbf{X}\mathbf{1}_{n\times 1} = \mathbf{0}_{n\times 1}\} \\ \mathbf{L} \in \mathcal{N}_- = \{\, \mathbf{X} \in \mathbb{R}^{n\times n} \mid x_{p,q} \le 0, p \ne q \,\} \end{array}} \quad \mathbf{L} \text{ is Laplacian of a graph}$$

- Solved with ADMM

Optimizing Spectral Diversity for Graph Signal Coarsening



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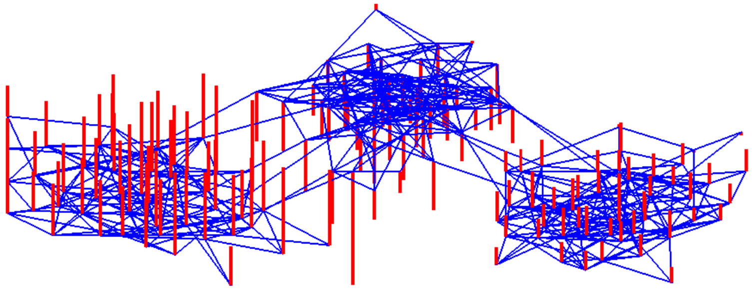
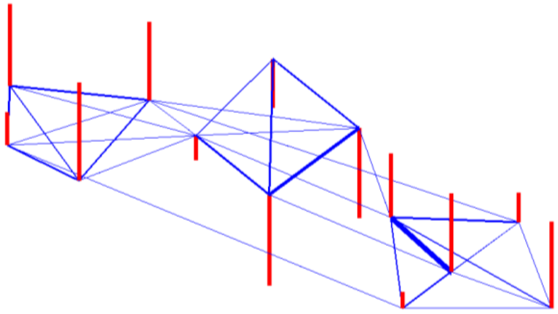
Experiments

- Original and coarsened graph signal in the vertex domain
  - From a 150-vertex graph with 3 communities
  - To a graph with 13 vertices



Original graph signal

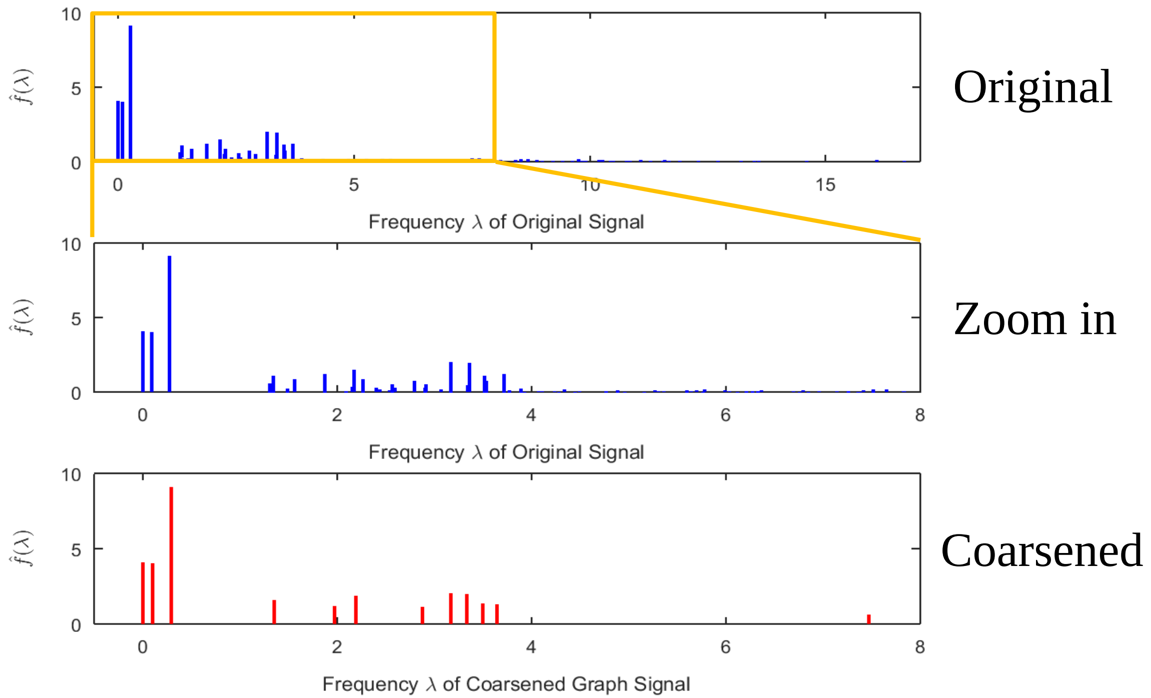


Coarsened graph signal

- Vertex domain similarity:
  - Preserve community structures
  - Preserve signal properties inside each community

Experiments

- Original and coarsened graph signal in the spectral domain



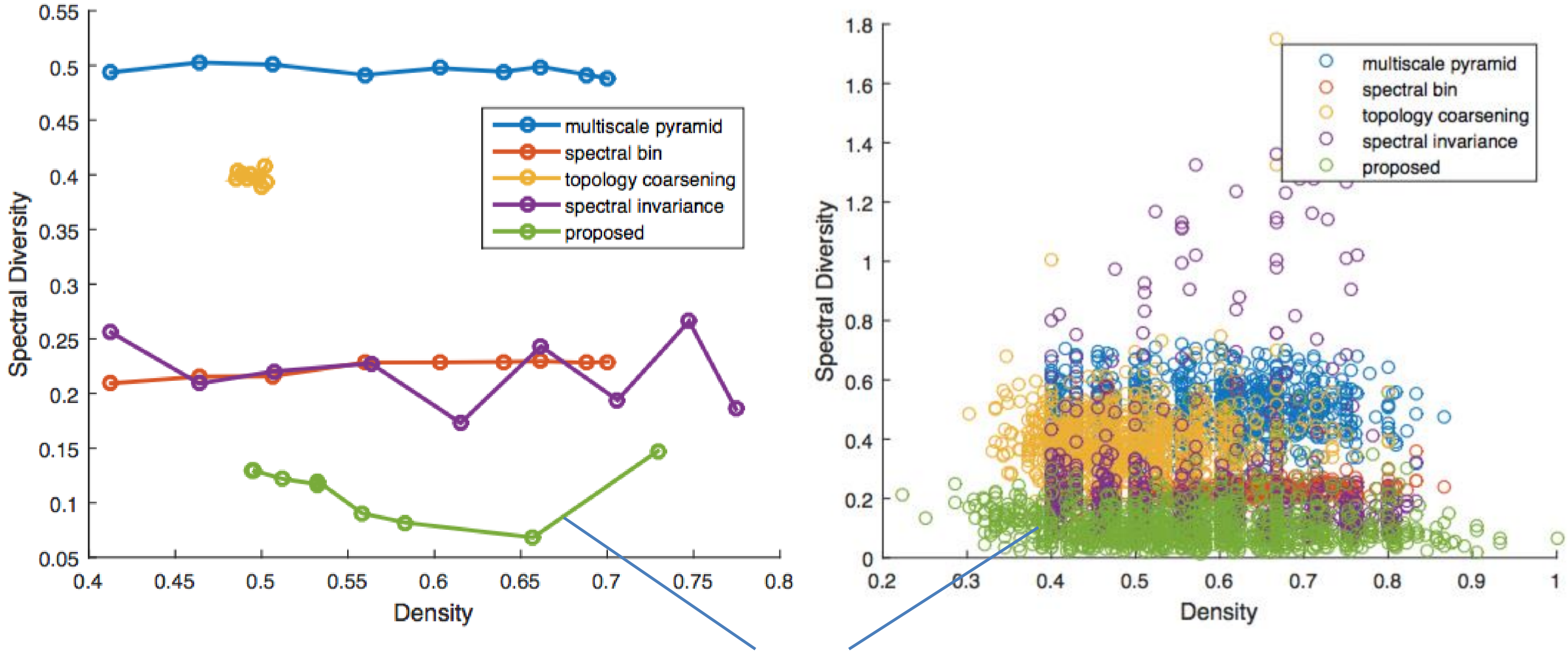
- Spectral domain similarity:
  - New graph spectrum is approximately a subset of that of the original one
  - Small spectral diversity between the original graph signal and the coarsened one

Comparisons

- Spectral diversity of original and coarsened graph signals when aiming for different density levels of graphs

Each point is the average of 100 trails

All trails



Proposed method performs best

Summary

- Spectral Diversity
  - Measure the similarity of variation speeds for signals across the graphs
  - Used for evaluating the quality of graph signal coarsening
- Optimizing Spectral Diversity for Graph Signal Coarsening
  - Achieve both vertex and spectral domain similarities between the original and coarsened graph signals

Thank you!

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