Super-Resolution DoA Estimation via Continuous Group Sparsity in The Covariance Domain

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Driven to Discover℠

Introduction

Directions-of-arrival (DoA) estimation — *Locating, with high resolution, closely-spaced DoAs with few snapshots.*

Conventional DoA estimators:

- Parametric methods
- Maximum Likelihood Estimator, MUSIC, ESPRIT, Matrix Pencil

Sparse model DoA estimator:

- Exploit sparsity in the model and discretize the search domain on grids
- Solve L_1 norm minimization problem
- Problem with off-grid DoAs

Continuous-domain viewpoint

- Use the super-resolution theory to provide a continuous-valued paramter gridless recovery method
- Solve a Total Variation norm minimization for a complex measure
- **Objective:** Promote group-sparsity in the super-resolution framework

System Model

DoA estimation problem — *Covariance model*

- *Single measurement vector* (SMV): K signals received by a linear array with M sensors, the observed measurement at time t is

$$\mathbf{y}(t) = \sum_{k=1}^{K} x_k(t)\mathbf{g}(\theta_k) + \mathbf{n}(t) = \mathbf{G}\mathbf{x}(t) + \mathbf{n}(t) \qquad (1)$$

  - Uncorrelated signal $x_k(t) \sim (0, \sigma_k^2)$
  - $\mathbf{g}(\theta_k) \in \mathbb{C}^{M\times 1}$ with m-th entry $e^{-j2\pi\frac{dm}{\lambda}sin\theta_k}$
  - *Multiple measurement vector* (MMV): $\mathbf{Y} = [\mathbf{y}(1), \ldots, \mathbf{y}(T)] = \mathbf{G}\mathbf{X} + \mathbf{N}, T > 1$

- The covariance matrix of observed vectors

$$\tilde{\mathbf{R}} = E[\mathbf{y}\mathbf{y}^H] = \sum_{k=1}^{K} \sigma_k^2 \mathbf{g}(\theta_k)\mathbf{g}(\theta_k)^H + \sigma^2\mathbf{I} \qquad (2)$$

- In reality, we compute $\mathbf{R} = \sum_{t=1}^{T} \mathbf{y}(t)\mathbf{y}(t)^H/T$ as

$$\mathbf{R} = \sum_{k=1}^{K} \sigma_k^2 \mathbf{g}(\theta_k)\mathbf{g}(\theta_k)^H + \mathbf{V}. \qquad (3)$$

Super-Resolution Theory

The super-resolution theory [1, 2]

- Consider a continuous signal $s(\tau), \tau \in [-1, 1]$ is

$$s(\tau) = \sum_{k=1}^{K} a_k \delta_{\tau_k}, \qquad (4)$$

  - a_k is complex-valued, and δ_{τ_k} is a Dirac measure at τ_k.
  - Denote data vector $\mathbf{s} = [a_1, \ldots, a_K]^T$.

Super-Resolution Theory

- The Fourier transform of $s(\tau)$ is

$$r(n) = \int_{-1}^{1} e^{-j2\pi n\tau} s(d\tau) = \sum_{k=1}^{K} a_k e^{-j2\pi n\tau_k}, n = -f_c, \ldots, f_c$$

- With arbitrary noise $\mathbf{e}$, we have $\mathbf{r} = \mathcal{F}s + \mathbf{e}$, where $\mathcal{F}$ denotes the linear operator to measure the $2f_c + 1$ lowest frequency coefficients.

Total Variation (TV) norm minimization

- For a complex meaure on a Borel set $B \in \mathcal{B}(\mathbb{T})$. TV norm is denoted by

$$\|s\|_{TV} = \sup \sum_{k=1}^{\infty} |s(B_k)| \qquad (5)$$

- Solve a convex optimization problem

$$\min_s \|s\|_{TV} \quad \text{s.t. } \|\mathcal{F}s - \mathbf{r}\|_2 \le \epsilon. \qquad (6)$$

The Proposed Method

Reformulation of the Spatial Covariance Model — *Recast the covariance model into a MMV-like one*

- Instead of vectorizing equation (3), we have

$$\mathbf{R} = [\mathbf{r}_0, \mathbf{r}_1, \ldots, \mathbf{r}_{M-1}] = \sigma_1^2\bar{\mathbf{G}}(\theta_1) + \cdots + \sigma_K^2\bar{\mathbf{G}}(\theta_K) + \mathbf{V},$$

where $\mathbf{g}(\theta_k)\mathbf{g}(\theta_k)^H = \bar{\mathbf{G}}(\theta_k)$ is a Toeplitz matrix expressed by $\bar{\mathbf{G}}(\theta_k) = [\mathbf{a}_0(\theta_k), \mathbf{a}_1(\theta_k), \ldots, \mathbf{a}_{M-1}(\theta_k)] \in \mathbb{C}^{M\times M}$.

- Then, we have

$$\mathbf{r}_l = \sigma_1^2\mathbf{a}_l(\theta_1) + \cdots + \sigma_K^2\mathbf{a}_l(\theta_K) + \mathbf{v}_l = \sum_k \sigma_k^2\mathbf{a}_l(\theta_k) + \mathbf{v}_l,$$
$$= \mathbf{A}_l\mathbf{p} + \mathbf{v}_l, \forall l = 0, \ldots, M-1 \qquad (7)$$

where $\mathbf{A}_l = [\mathbf{a}_l(\theta_1), \ldots, \mathbf{a}_l(\theta_K)] \in \mathbb{C}^{M\times K}$, $\mathbf{p} = [\sigma_1^2, \ldots, \sigma_K^2]^T \in \mathbb{R}^{K\times 1}$.

- Thus, $\mathbf{R}$ is rewritten as

$$\mathbf{R} = [\mathbf{A}_0\mathbf{p}, \mathbf{A}_1\mathbf{p}, \ldots, \mathbf{A}_{M-1}\mathbf{p}] + \mathbf{V}, \qquad (8)$$

- In ULA, $\mathbf{a}_l(\theta_k) = [e^{-j(-l)\xi_k}, \ldots, e^{-j(M-1-l)\xi_k}]^T \in \mathbb{C}^{M\times 1}$, $\forall l = 0, \ldots, M-1$, where $\xi_k = \frac{d}{\lambda}2\pi sin\theta_k$.

Continuous Group-Sparsity

Extend the SR theory from SMV to MMV-like system

- Extend a continuous signal into the MMV space by

$$s(\tau; t) = \sum_{k=1}^{K} b_{kt}\delta_{\tau_k}, t = 1, \ldots, T \qquad (9)$$

  - b_{kt} is complex-valued at time t
  - Denote $\mathcal{T} = \{\tau_k\}_{k=1}^{K}$ as the support set.
  - Denote $\mathbf{S} = [\mathbf{s}_1, \ldots, \mathbf{s}_T]$ where $\mathbf{s}_t = [b_{1t}, \ldots, b_{Kt}]^T$.

Continuous Group-Sparsity

- Similarly with noise $\mathbf{e}^t$, and by FT, we have

$$\mathbf{r}_{sr}^t = \mathcal{F}s(\tau; t) + \mathbf{e}^t, \forall t = 1, \ldots, T \qquad (10)$$

Block Total Variation (BTV) norm

- By using multiple measurements for a complex meaure, we denote

$$\|s\|_{TV,p} = \sup \sum_{k=1}^{\infty} \|s(B_k; :)\|_p. \qquad (11)$$

$\|s(B_k; :)\|_p = (\sum_{t=1}^{T} |s(B_k; t)|^p)^{1/p}$ and $s(B_k; t) = b_{k,t}$.

- $\min \|s\|_{TV,p} \iff \min \|\mathbf{S}\|_{1,p} = \sum_k \|\mathbf{S}_{k,:}\|_p$

BTV-norm Minimization

Fit DoA estimation problem in the group-sparsity framework:

Letting $\tau_k = sin(\theta_k)$, $t = l$, $T = M - 1$, and $f_c = (M-1)/2$, we have

$$\mathbf{r}_{sr}^l = \mathcal{F}_l s(\tau; l) + \mathbf{e}^l = \mathbf{A}_l\mathbf{p} + \mathbf{v}_l = \mathbf{r}_l, \ l = 0, \ldots, M-1.$$

- Propose the BTV norm minimization problem

$$\min_s \|s\|_{TV,1} \quad \text{s.t.} \sum_{l=0}^{M-1} \|\mathcal{F}_l s - \mathbf{r}_l\|_2 \le \epsilon. \qquad (12)$$

Theorem 1 extended from [2]. *Let $\mathcal{T} = \{\tau_k\}_{k=1}^{K}$ as the support set. If the minimum distance $\Delta(\boldsymbol{\theta})$ obeys*

$$\Delta(\boldsymbol{\theta}) = \inf_{\tau_i, \tau_j \in \mathbb{T}} |\tau_i - \tau_j| \ge \frac{4}{f_c}\frac{\lambda}{d},$$

then the high resolution detail of continuous signal s can be recovered with high probability by solving block total variation norm minimization problem (12).

- To estimate the support set, we derive the dual form of (12)

$$\max_{\mathbf{U}} \text{Re}\{<\mathbf{R}, \mathbf{U}>\} - \epsilon\|\mathbf{U}\|_F \qquad (13)$$

$$\text{s.t.} \begin{bmatrix} \mathbf{Q}^l & \mathbf{u}_l \\ \mathbf{u}_l^H & 1 \end{bmatrix} \succeq 0, \forall l = 0, \ldots, M-1$$

$$\sum_{i=1}^{M-j} \mathbf{Q}_{i,i+j}^l = \begin{cases} 1, & j = 0, \\ 0, & j = 1, 2, \ldots, M-1 \end{cases},$$

where $\mathbf{Q}^l \in \mathbb{C}^{M\times M}$ is a Hermitian matrix, $\forall l$.

Lemma 2 *Let s_{est} and $\mathbf{u}_{l,est}$ be a pair of primal-dual solutions to (12) and (13). Then*

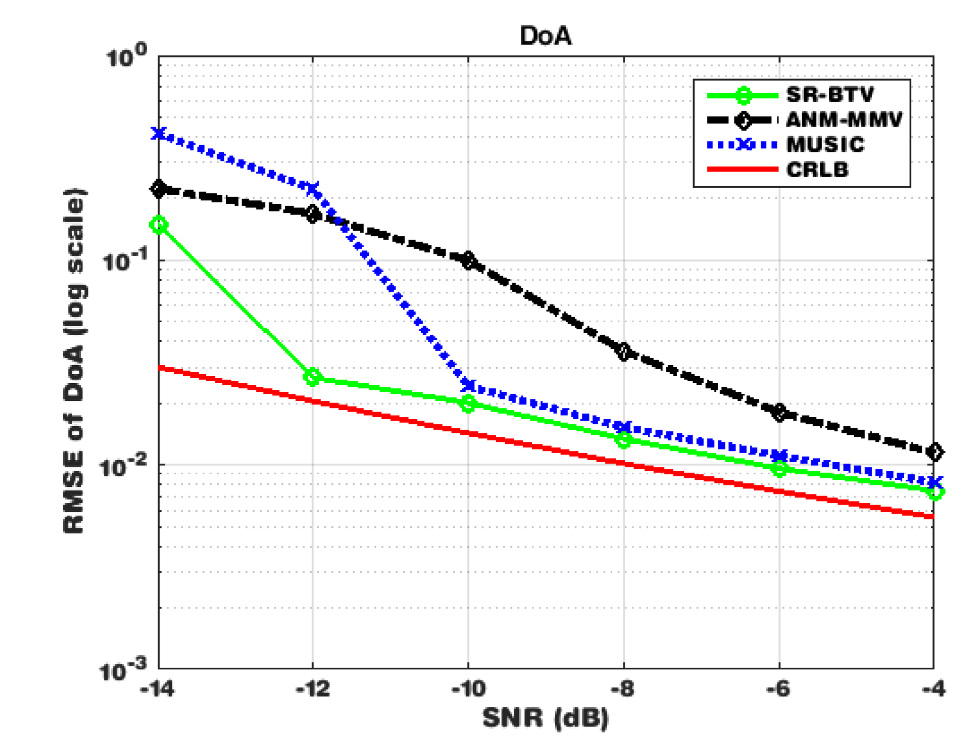
$$(\mathcal{F}_l^*\mathbf{u}_{l,est})(\tau) = sign(s_{est}(\tau; l)), \forall \tau \in \mathbb{T} \text{ s.t. } s_{est}(\tau; l) \ne 0.$$

- Perform the root finding on the $|(\mathcal{F}_l^*\mathbf{u}_{l,est})(\tau)|^2 = 1$, $\forall l$, to get the estimated support sets $\mathcal{T}_{est}^l = \{\tau_{k,est}^l\}_{k=1}^{K}$ and its union set $\mathcal{T}_{est} = \bigcup_l \mathcal{T}_{est}^l$.
- Obtaining $\mathbf{G}_{est}$ by $\mathcal{T}_{est}$, we solve

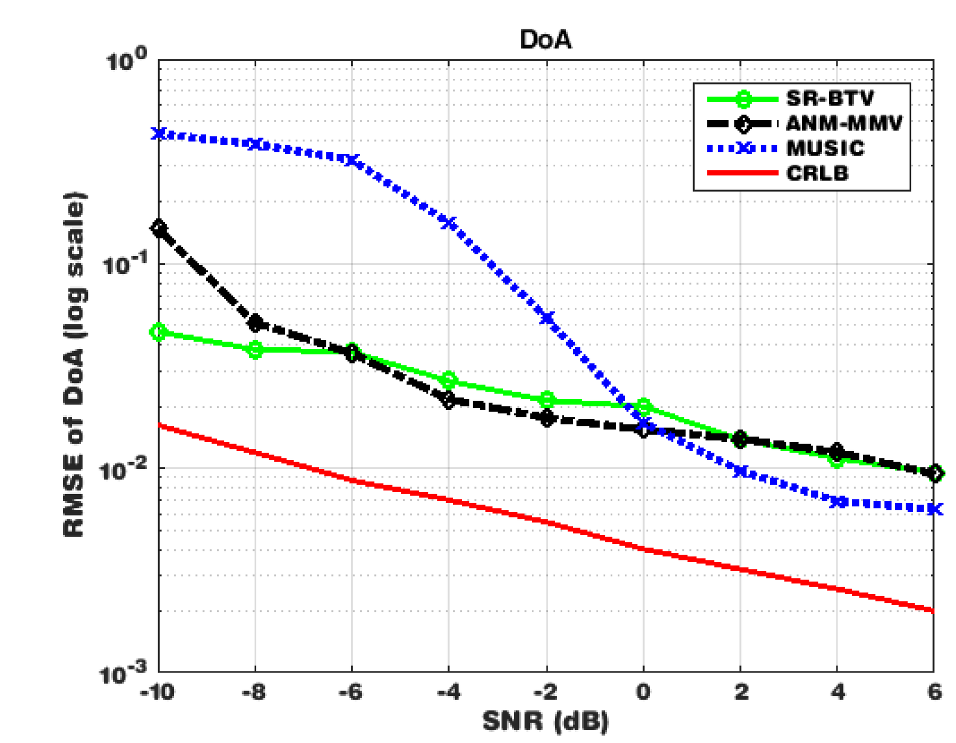
$$\hat{\mathbf{X}} = \arg\min_{\mathbf{X}} \frac{1}{2}||\mathbf{Y} - \mathbf{G}_{est}\mathbf{X}||_F^2 + \gamma||\mathbf{X}||_{2,1},$$

where $||\mathbf{X}||_{2,1} = \sum_{k=1}^{|\mathcal{T}_{est}|} \|\mathbf{X}_{k,:}\|_2$, and $\mathbf{X}_{k,:}$ denotes the k^{th} row of $\mathbf{X}$.

Numerical Results



(a) RMSE of DoA estimation vs SNR for the case of uncorrelated sources.



(b) RMSE of DoA estimation vs SNR for the case of correlated sources.

ULA of 9 sensors, 2 sources with DoA $sin(\boldsymbol{\theta}) = [0.2165251, 0.4665251]$, correlation coefficient$= 0.9$, $T = 100$

Summary

- Reformulated the covariance model.
- Proposed an BTV norm minimization.
- Robust performance of SR-BTV compared wtih MUSIC and ANM-MMV [3] in cases of uncorrelated and correlated sources.

Bibliography

[1] E. J. Candès and C. Fernandez-Granda, "Super-resolution from noisy data," *Journal of Fourier Analysis and Applications*, vol. 19, no. 6, pp. 1229–1254, 2013.

[2] ——, "Towards a mathematical theory of super-resolution," *Communications on Pure and Applied Mathematics*, vol. 67, no. 6, pp. 906–956, 2014.

[3] Z. Yang and L. Xie, "Exact joint sparse frequency recovery via optimization methods," *arXiv preprint arXiv:1405.6585*, 2014.