Perfect sequences are sequences with zero autocorrelation. A Perfect Gaussian sequence is a perfect sequence that each value in the sequence is a Gaussian integer. Considering the sequence length, there are three cases: $N$ is prime, $N$ is a power of prime $p^m$ or $N = pq$, where $p$ and $q$ are coprime but not necessary prime numbers.

**Introduction**

Perfect sequences are sequences with zero autocorrelation (ZAC). For constant $C$,

$$\sum_{n=0}^{N-1} x^n(m) x(n) = C\delta(m)$$

for some constant $C$.

When considering whether a sequence if ZAC or not, a smart way is to calculate its discrete Fourier transform (DFT). Benedetto[4] has proven that a sequence is ZAC if and only if its DFT is constant amplitude (CA).

$$|X(m)| = A$$

for some constant $A$.

When the sequence length is prime or power of prime, we can use Legendre symbol to construct perfect Gaussian integer sequences. Legendre symbol or sequence[LS][11, 12] has a strong connection to quadratic Gauss sum, which is also a summation of complex roots of unity, like Ramanujan’s sum. For composite number, we propose some novel methods to construct perfect Gaussian integer sequences, from the naive zero padding and convolution, to decomposing $N$ into different groups.

**Objectives**

To construct perfect Gaussian integer sequences of arbitrary length $N$:

- Perfect sequences are sequences with zero autocorrelation.
- Gaussian integer is a number in the form $a + bi$ where $a$ and $b$ are integer.
- A Perfect Gaussian sequence is a perfect sequence that each value in the sequence is a Gaussian integer.
- Considering the sequence length, there are three cases: $N$ is prime, $N$ is a power of prime $p^m$ or $N = pq$, where $p$ and $q$ are coprime but not necessary prime numbers.

**Perfect Gaussian Integer Sequences of Arbitrary Length**

If $N = p^m$ or $N = p^m$ using Legendre sequence and Gauss sum:

Legendre symbol is defined as

$$\left( \frac{n}{N} \right) =
\begin{cases}
1, & \text{if } 2x, x^2 \equiv n \pmod{N} \\
0, & \text{if } n \equiv 0 \pmod{N} \\
-1, & \text{otherwise.}
\end{cases}$$

And the Gauss sum is defined as

$$G(k) = \frac{N-1}{\pi i k} \sum_{n=0}^{N-1} e^{-2\pi i kn/N}$$

A well-known result[5] is that

$$G(k) = \begin{cases}
\frac{1}{\sqrt{N}}, & N \equiv 1 \pmod{4} \\
-\frac{i}{\sqrt{N}}, & N \equiv 3 \pmod{4}
\end{cases}$$

In other words, the Fourier transform of Legendre Sequences is almost CA, with the only exception on $k = 0$, the first point. The amplitude is $\sqrt{N}$ thus our goal is to find a Gaussian integer $a$ and some integers $b$ and $c$ such that

$$|a|^2 = b^2 + Nc^2$$

Then a sequence that

$$f(n) = \begin{cases}
a, & n = 0 \\
\sqrt{N}c + bi, & n \neq 0
\end{cases}$$

is CA, and the DFT of $f(n)$ is ZAC in Gaussian integer.

**Example**

$$f(n) = \{6 + 1i, 5i + 2\sqrt{3}, 5i - 2\sqrt{3}\}$$

$$F(k) = \{6 + 11i, 6 - 10i, 6 + 2i\}$$

Let $N = 3^2$, and $c=2,b=5,a=6+i$, since $d(n)$ is

$$d(n) = \{0, 1, -1, 0, 1, -1, 0, 1\}$$

The first iteration gives us

$$\{0, 2\sqrt{3} + 5i, -2\sqrt{3} + 5i, 0, 2\sqrt{3} + 5i, -2\sqrt{3} + 5i, 0, 2\sqrt{3} + 5i, -2\sqrt{3} + 5i\}$$

Now second eliminates the remaining zeros.

$$f(n) = \{6 + 1i, 2\sqrt{3} + 5i, -2\sqrt{3} + 5i, 1 - 6i, 2\sqrt{3} + 5i, -2\sqrt{3} + 5i, 1 - 6i, 2\sqrt{3} + 5i, -2\sqrt{3} + 5i\}$$

$$F(k) = \{8 + 19i, 5 + 7i, 5 + 7i, 8 - 44i, 5 + 7i, 5 + 7i, 8 - 4i, 5 + 7i, 5 + 7i\}$$

which is CA, and its Fourier transform

$$\{6 + 43i, 6 + 4i, 6 + 16i, 21 + 13i, 21 - 41i, 21 + 31i, 21 - 13i, 21 + 31i, 21 - 41i, 21 + 31i\}$$

is a perfect Gaussian integer sequence.

**N = pq where p and q are coprime**

Simple zero paddling method:

1. Take a ZAC from $p$ and $q$.
2. Interpolate $q-1$ and $p-1$ zeros to these signals to get two signals of length $N$.
3. Convolution these two signals, then we get a ZAC.

Using the idea of prime-factor algorithm:

Recall that DFT of size $N = N_1 N_2$ can be done by taking DFT of size $N_1$ and $N_2$ separately[14]. To construct perfect sequence, just

1. Divide $n = 0, 1, 2, ..., N-1$ into groups $S_k$, where

$$S_k = \{n | gcd(n, N) = d\}$$

2. For each group, use LS or GLS to ensure CA
3. Take DFT

**Example**

The factors of 15 is 1, 3, 5, 15. So we divide our signal into 4 groups

$$f(0) = \{0, 0, 0, 0\}$$

$$f(5) = \{f(0), f(0), f(0), f(0)\}$$

$$f(10) = \{0, 0, 0, 0\}$$

One possible outcome is by

$$61 = 6^2 + 5^2 = 1^2 + 2^2 + 15^2 + 2^2 + 3^2 = 4^2 + 3^2 \times 5$$

**Conclusion**

We propose several methods to generate zero autocorrelation sequences in Gaussian integer. If the sequence length is prime number, we can use Legendre symbol and provide more degree of freedom than Yang’s method. If the sequence is composite, we develop a general method to construct ZAC sequences. Zero padding is one of the special cases of this method, and it is very easy to implement.

**References**


